Coalitional Manipulation on Communication Network

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Abstract

In an abstract model of division problems, we study division rules that are not manipulable through a reallocation of individual characteristic vectors within a coalition (e.g. reallocation of claims in bankruptcy problems). A coalition can be formed under a given communication network, a (non-directed) graph, if members of this coalition are connected on the graph. We offer a characterization of non-manipulable division rules without any assumption on the graph structure. When the graph is complete, this result reduces to the results established by previous authors. We also consider other special cases such as trees and graphs without a "bridge". The family of *reallocation-proof* rules can get larger or smaller depending on the graph structure. Our abstract model can have various special examples such as bankruptcy, surplus sharing, cost sharing, income redistribution, social choice with transferable utility, etc.

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1 Introduction

Division problems often take the following abstract form. There are a finite number of agents. Each agent is characterized by a vector in $\mathbb{R}^K_+$, where K is the set of characteristics. An amount of resource, a real number, has to be divided among these agents. A systematic method of division can be described by a *division rule* associating with each division problem a vector of individual shares, or awards.

A number of earlier authors have studied division rules that are robust to coalitional manipulation through a reallocation of characteristic vectors (reallocation of claims among a group of investors in the context of bankruptcy; reallocation of contributions in the context of surplus sharing, etc.). O'Neill (1982), Moulin (1985a, 1987), Chun (1988), Moulin and Shenker (1992), de Frutos (1999), Ching and Kakker (2001), and Ju (2003) consider specialized models dealing with bankruptcy (or taxation), surplus sharing, social choice with transferable utility, and cost allocation. Ju, Miyagawa, and Sakai (2003) consider the same abstract model as ours. One common and crucial assumption in these works is that agents can form a coalition without any restriction. In this paper, we consider more realistic scenario, in which coalition formation is subject to communication network.

A communication network is described by a (non-directed) graph. A coalition can be formed under a graph if members of this coalition are connected. A division rule satisfies *reallocation-proofness* if no coalition can increase the total award by a reallocation of characteristic vectors among its members. Our main result is a characterization of *reallocation-proof* division rules. It is established without any assumption on the graph. When the graph is complete (any two nodes are directly linked), this result reduces to the results established by previous authors. We also consider other special cases such as trees and rigid graphs (graphs without a "bridge"). The family of *reallocation-proof* rules can get larger or smaller depending on the graph structure.

The rest of the paper is organized as follows. In Section 2, we define our model, communication network, axioms, and some important rules in this paper. In Section 3, we state and prove preliminary results. In Section 4, we state our main result. Some proofs are in Appendices A-C.

2 Definitions

2.1 Model

There is a finite set $N = \{1, 2, \dots, n\}$ of *agents*. Each agent $i \in N$ is characterized by a vector $c_i \equiv (c_{ik})_{k\in K} \in \mathbb{R}^K_+$, where K denotes the set of issues. We refer to c_i as i's *characteristic vector*. A profile of characteristic vectors of agents is denoted by $c \equiv (c_i)_{i\in N} \in \mathbb{R}^{N\times K}_+$, and the sum of these vectors is denoted by

$$\bar{c} \equiv (\bar{c}_k)_{k\in K} \equiv (\sum_{i\in N} c_{ik})_{k\in K} \in \mathbb{R}^K_+.$$

A *problem* is a pair $(c, E) \in \mathbb{R}^{N\times K}_+ \times \mathbb{R}_{++}$, where $c \in \mathbb{R}^{N\times K}_+$ is a profile of characteristic vectors and $E \in \mathbb{R}_{++}$ is an amount to be divided. For simplicity, we only consider problems such that $\bar{c}_k > 0$ for each $k \in K$. A *domain* is a non-empty set of problems and is denoted by $\mathcal{D}$. A division rule, or briefly, a *rule* over a domain $\mathcal{D}$ is a function f associating with each problem $(c, E) \in \mathcal{D}$ a vector of awards $f(c, E) \in \mathbb{R}^N$. A domain $\mathcal{D}$ is *rich* if, for each problem $(c, E) \in \mathcal{D}$ and each profile $\bar{c} \in \mathbb{R}^{N\times K}_+$ such that $\bar{c}' = \bar{c}$, we have $(\bar{c}', E) \in \mathcal{D}$. That is, $\mathcal{D}$ is rich if it is closed under reallocations of characteristic vectors. We restrict our attention to rich domains. For each problem $(c, E) \in \mathcal{D}$, let

$$\mathcal{D}(\bar{c}, E) \equiv \{(c', E) \in \mathbb{R}^{N\times K}_+ \times \mathbb{R}_{++} : \bar{c}' = \bar{c}\}.$$

Then richness says that, for each $(c, E) \in \mathcal{D}$, we have $\mathcal{D}(\bar{c}, E) \subseteq \mathcal{D}$. Examples of rich domains are the set of bankruptcy problems in O'Neill (1982), the set of surplus sharing problems in Moulin (1987), the set of social choice problems with transferable utilities in Moulin (1985), the set of cost sharing problems in Moulin and Shenker (1992), etc.

We also use the following additional notation. For each $S \subseteq N$ and each $c \in \mathbb{R}^{N\times K}_+$,

$$\bar{c}_S \equiv (\bar{c}_{Sk})_{k\in K} \equiv (\sum_{i\in S} c_{ik})_{k\in K} \in \mathbb{R}^K_+.$$

Similarly, for each $S \subseteq N$ and each $x \in \mathbb{R}^N_+$,

$$\bar{x}_S \equiv \sum_{i\in S} x_i.$$

Given $x, y \in \mathbb{R}^M$, $x \geqq y$ means that $x_m \geq y_m$ for each m; $x \geq y$ means that $x \geqq y$ and $x \neq y$; and $x > y$ means that $x_m > y_m$ for each m.

2.2 Communication Network and Coalition Structure

Before defining "coalitional manipulation", we first need to explain possible coalition formations. We assume that agents form a coalition through communication on a network. The communication network is fixed throughout the paper. It is described by a (non-directed) *graph* consisting of a set of *nodes* N and a set of *edges* $D \equiv \{\{i,j\} : i,j \in N \text{ and } i \neq j\}$. Let $G \equiv (N, D)$. For simplicity, we sometimes denote an edge $\{i,j\} \in D$ by ij. Two nodes, i and j, are *adjacent* if ij is an edge.

A *complete graph* is a graph $G \equiv (N, D)$ such that for each $i, j \in N$ with $i \neq j$, $ij \in D$. A *path* is a sequence of edges which are successively intersecting. A path is denoted simply by listing nodes that the path follows. A *line* is a path that never passes a node more than once. For each $h, i, j \in N$, we say i is *between* h *and* j if every path including h and j includes also i. A *cycle* is a path that passes more than two nodes and that passes one and only one node twice. With a slight abuse of terminology, we say that a graph is a *cycle* when the graph itself is a cycle. Similarly, we say that a graph is a *line*. A *total line* is a line containing all nodes in N. A *total cycle* is a cycle containing all nodes in N.

For each $S \subseteq N$, let $G_S \equiv (S, D_S \equiv \{ij \in D : i,j \in S\})$ be the *subgraph on* S. We say a subgraph G_S is *connected* if for any two nodes $i, j \in S$, there is a path in G_S from i to j. Note that when $S = \emptyset$ or a singleton, G_S is connected trivially. We say that S is *connected* when G_S is connected. Coalition S is *admissible* if S is connected. Let $\mathcal{C}(G)$ be the set of admissible coalitions, called, the *coalition structure* on G. For example, when G is a *complete graph*, $\mathcal{C}(G)$ equals the set of all subsets of N, that is, 2^N, which is called the *unrestricted coalition structure.*

Throughout the paper, we assume that G *is connected.* However, our results are easily extended to the general case.[1]

A *tree* is a connected graph in which every two nodes have one and only one path from one to another. A node i in a tree is an *end node* if i is not between any two other nodes, that is, for all $h, j \in N \backslash \{i\}$, i is not between h and j. If G is a tree, by choosing any node $i^* \in N$ as a *root*, we can define the *directed tree with root* i^*, denoted by $G(i^*)$. In the directed tree $G(i^*)$, for each $i \in N$, let $s(i)$ be the set of *successors* of i, including i itself, and $s^0(i)$ the set of successors of i, not including i. Let $p(i)$ be the set of *predecessors*, including i itself, and $p^0(i)$ the set of predecessors of i, not including i. Let $sm(i)$ be the set of *immediate successors*

[1] Note that any (possibly disconnected) graph is partitioned into the unique family of maximal connected subgraphs. Our results can be applied for each of these maximal connected subgraphs.

of i and $pm(i)$ the *immediate predecessor* of i. Clearly, $j \in sm(i)$ if and only if $i \in pm(j)$. It should be noted that all these functions, $s(\cdot)$, $s^0(\cdot)$,$sm(\cdot)$, $p(\cdot)$, $p^0(\cdot)$, and $pm(\cdot)$, depend on the choice of the root i^*.

An edge $ij \in D$ is called a *connection edge* (also called an "isthmus" or a "bridge" in Wilson 1979) if deleting ij from D results in a disconnected graph, that is, $(N, D\backslash\{ij\})$ is not connected. A graph G is *rigid* if it has no connection edge.[2] Thus a rigid graph remains connected after deleting any one of its edges. We next define graphs in which no single node plays a critical role in keeping the graph connected. A node $i \in N$ is called a *connection node* if deleting i from G results in a disconnected subgraph of G, that is, $G_{N\backslash\{i\}}$ is not connected. A graph G is *rigid** if it is connected and it has no connection node.[3] Thus a rigid* graph stays connected after a deletion of any single node.

2.3 Axioms

Our main objective is to study rules that are robust to coalitional manipulations through reallocations of characteristic vectors. Since coalition formation is constrained by a graph, such a robustness can be formalized by the requirement that the total amount allocated to each admissible coalition $S \in \mathcal{C}(G)$ should not be affected by any reallocation of c_i's within S. Formally:

Reallocation-Proofness. For each $(c, E) \in \mathcal{D}^N$, each $S \in \mathcal{C}(G)$, and each $c' \in \mathbb{R}_+^{S\times K}$, if $\bar{c}'_S = \bar{c}_S$,

$$\sum_{i\in S} f_i(c', c_{N\backslash S}, E) = \sum_{i\in S} f_i(c, E). \qquad (1)$$

This axiom has been introduced by Moulin (1985a) and Chun (1988) in the contexts of social choice with transferable utilities and claims problems, respectively (they call this axiom "no advantageous reallocation").

In the context of claims problems and their variants, the axiom means that no group of agents can change their aggregate share by reallocating claims within the group. If the left-hand side of (1) is larger than the right-hand side, then group S with claim profile $(c_i)_{i\in S}$ can gain by reallocating their claims to c'_S (and making appropriate side-payments). If the reverse inequality holds, then group S with claims $(c'_i)_{i\in S}$ can gain.

[2] Thus a graph is rigid if and only if its degree of "edge connectivity" (see p. 29 of Wilson 1979 for the definition) is equal to 1.

[3] Thus a graph is rigid* if and only if its degree of "connectivity" (see p. 29 of Wilson 1979 for the definition) is equal to 1.

We also consider a weaker condition, by focusing on coalitions by pairs.

Pairwise Reallocation-Proofness. For each $(c, E) \in \mathcal{D}^N$, each $ij \in D$ (so $\{i, j\} \in \mathcal{C}(G)$) and each $c'_i, c'_j \in \mathbb{R}_+^{S \times K}$, if $c'_i + c'_j = c_i + c_j$,

$$f_i(c'_i, c'_j, c_{N \setminus \{i,j\}}, E) + f_j(c'_i, c'_j, c_{N \setminus \{i,j\}}, E) = f_i(c, E) + f_j(c, E).$$

The next axiom is a useful implication of *reallocation-proofness* (see Lemma 2). It says that any admissible coalition cannot change, through a reallocation of characteristic vectors, the shares of others, without affecting its own aggregate share. This axiom is similar, in spirit, to "non-bossiness" in economic environments introduced by Satterthwaite and Sonnenschein (1981).

Non-Bossiness. For each $(c, E) \in \mathcal{D}^N$, each $S \in \mathcal{C}(G)$, and each $c' \in \mathbb{R}_+^{N \times K}$, if $\bar{c}'_S = \bar{c}_S$, $c'_{N \setminus S} = c_{N \setminus S}$, and $\sum_{i \in S} f_i(c', E) = \sum_{i \in S} f_i(c, E)$,

$$f_{N \setminus S}(c', E) = f_{N \setminus S}(c, E). \tag{2}$$

The next axiom is the pairwise version of *non-bossiness*.

Pairwise Non-Bossiness. For each $(c, E) \in \mathcal{D}^N$, each $ij \in D$, and each $c'_i, c'_j \in \mathbb{R}_+^{S \times K}$, if $c'_i + c'_j = c_i + c_j$ and $f_i(c'_i, c'_j, c_{N \setminus \{i,j\}}, E) + f_j(c'_i, c'_j, c_{N \setminus \{i,j\}}, E) = f_i(c, E) + f_j(c, E)$,

$$f_{N \setminus \{i,j\}}\left(c'_i, c'_j, c_{N \setminus \{i,j\}}, E\right) = f_{N \setminus \{i,j\}}(c, E).$$

For example, in the context of bankruptcy problems, there is a large family of *non-bossy* rules, known as "parametric rules".

In some of our results, we characterize rules satisfying some combinations of the following axioms as well as *reallocation-proofness*.

The next axiom says that awards should add up to the amount to divide:

Efficiency. For each $(c, E) \in \mathcal{D}$, $\sum_{i \in N} f_i(c, E) = E$.

Note that on the compact set $\mathcal{D}(\bar{c}, E)$, each agent's characteristic vector is both bounded above and below. Then, it is appealing to require that each agent should not get unlimited reward or unlimited loss on the set $\mathcal{D}(\bar{c}, E)$. The next axiom states an even weaker condition that at least one agent's award should be bounded above or below on $\mathcal{D}(\bar{c}, E)$.

One-Sided Boundedness. For each $(c, E) \in \mathcal{D}$, there exists $i \in N$ such that $f_i(\cdot, E)$ is bounded from either above or below over $\mathcal{D}(\bar{c}, E)$.

This axiom is implied by each of the following two axioms. The first one requires awards to be non-negative:

Non-Negativity. For each $(c, E) \in \mathcal{D}$ and each $i \in N$, $f_i(c, E) \geq 0$.

Another axiom that implies *one-sided boundedness* is *no transfer paradox* (Moulin 1985a). It says that no agent can increase its award by transferring part of its characteristic vector to other agents:

No Transfer Paradox. For each $(c, E) \in \mathcal{D}$, each $c' \in \mathbb{R}_+^{N \times K}$, each $i, j \in N$ with $i \neq j$, and each $t \in [0, c_i] \subseteq \mathbb{R}_+^K$,[4]

$$f_i(c_i - t, c_j + t, c_{-\{i,j\}}, E) \leq f_i(c_i, c_j, c_{-\{i,j\}}, E).$$

The next axiom says that no amount should be awarded to agents with the zero characteristic vector:

No Award for Null. For each $(c, E) \in \mathcal{D}$ and each $i \in N$, if $c_i = 0$, then $f_i(c, E) = 0$.

2.4 Examples of Division Rules

For the case when characteristic vectors are single-dimensional (i.e., $|K| = 1$), one of the simplest and best-known rules is proportional rule, which divides the total amount proportionally to characteristic vectors.

Definition 1 (Proportional Rule, $|K| = 1$). For each $(c, E) \in \mathcal{D}$ and each $i \in N$,[5]

$$f_i(c, E) = \frac{c_i}{\bar{c}} E.$$

Ju, Miyagawa, and Sakai (2003) extend the definition of proportional rule to the case of multi-dimensional characteristics, $|K| \geq 2$. A *weight function* is a function mapping each $(\bar{c}, E) \in \mathbb{R}_{++}^K \times \mathbb{R}_{++}$ into a weight vector in $\Delta^{|K|-1}$, $W \colon \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \Delta^{|K|-1}$.

Definition 2 (Proportional Rules, $|K| \geq 1$). A rule f is a *proportional rule* if there exists a weight function W such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = \sum_{k \in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E) E.$$

We use P^W to denote the proportional rule associated with W.

Note that P^W first applies the proportional rule to each single-dimensional sub-problem (c^k, E), where $c^k \equiv (c_{ik})_{i \in N}$, and then takes the weighted average

[4] Let $[0, c_i] \equiv [0, c_{i1}] \times \cdots \times [0, c_{iK}]$.

[5] The right-hand side is well-defined since we rule out problems for which $\bar{c} = 0$.

of the solutions to the sub-problems using the vector of weights $W(\bar{c}, E)$. The weights depend on the problem being considered but depend only on $(\bar{c}, E)$. Proportional rules are *efficient* since $\sum_{k\in K} W_k(\bar{c}, E) = 1$. Proportional rules also satisfy all other axioms defined in Section 2.3. It is evident that, if $|K| = 1$, Definition 2 reduces to Definition 1.

We now define *generalized proportional rules*, introduced by Ju, Miyagawa, and Sakai (2003). These rules are characterized by two functions $A\colon \mathbb{R}^K_{++}\times\mathbb{R}_{++} \to \mathbb{R}^N$ and $W\colon \mathbb{R}^K_{++}\times\mathbb{R}_{++} \to \mathbb{R}^K$, and i's award is given by the sum of the following two terms. The first term is $A_i(\bar{c}, E)$, which is independent of i's characteristic vector but may treat i differently from others. The second term is proportional to i's characteristic vector and treats agents symmetrically. On the other hand, the second term may treat issues asymmetrically, and the degree of importance attached to each issue $k \in K$ is given by $W_k(\bar{c}, E)$. Formally,

Definition 3 (Generalized Proportional Rules). There exist two functions $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $W\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = A_i(\bar{c}, E) + \sum_{k\in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E)E. \tag{3}$$

Note that W is not required to be a weight function, i.e., neither $W_k(\bar{c}, E) \geq 0$ nor $\sum_{k\in K} W_k(\bar{c}, E) = 1$ is required. Proportional rules are special cases where $A_i = 0$ and W is a weight function. Since, given $(\bar{c}, E)$, the second term of (3) is linear in c_{ik}, generalized proportional rules satisfy *reallocation-proofness* and *one-sided boundedness*. These rules do not necessarily satisfy other axioms in Section 2.3. Necessary and sufficient conditions for (A, W) to satisfy each of those axioms are stated in Proposition 2.

3 Preliminary Results

We first establish two useful lemmas. The first lemma shows that any reallocation of characteristic vectors among agents in a connected coalition can be described by successive reallocations among edges in this coalition.

Lemma 1. *If S is connected and $c, c' \in \mathbb{R}^{N\times K}_+$ are such that $\bar{c}'_S = \bar{c}_S$ and $c'_{N\setminus S} = c_{N\setminus S}$, then c' can be reached from c through successive reallocations of characteristic vectors among edges in S, that is, there exist a number r and $S_1, \cdots, S_r \in D_S$ and $c^1, c^2, \cdots, c^r \in \mathbb{R}^{N\times K}_+$ such that $\bar{c}^1_{S_1} = \bar{c}_{S_1}$, $c^1_{N\setminus S_1} = c_{N\setminus S_1}$, $c^r = c'$, and for each $m = 2, \cdots, r$, $\bar{c}^m_{S_m} = \bar{c}^{m-1}_{S_m}$ and $c^m_{N\setminus S_m} = c^{m-1}_{N\setminus S_m}$.*

Proof. Let S and $c, c' \in \mathbb{R}_+^{N\times K}$ be given as above. We give the idea of the proof and omit the formal proof that is tedious. Pick an agent, say 1, in S. For any $i \in S$, since S is connected, there is a path from i to 1, denoted by p_i, and we can move all i's characteristics in c_i to 1's through this path. Then we end up with $c'' \in \mathbb{R}_+^{N\times K}$ such that $c''_1 \equiv \bar{c}_S$, $c''_{S\backslash\{1\}} = 0$, and $c''_{N\backslash S} = c_{N\backslash S}$. Now we do the reverse changes, that is, for each $i \in S$, we use path p_i to increase i's vector from 0 to c'_i and decrease 1's vector from $\bar{c}_S$ to $\bar{c}_S - c'_i$. Throughout this procedure, we always have non-negative characteristic vectors for all agents and the constant sum of characteristic vectors of agents in S. At the end, we reach c'_S. Since there is no change made in the characteristic vectors of agents in $N\backslash S$, this is what we wanted to show. ∎

We now establish logical relation among *reallocation-proofness*, *non-bossiness*, and their *pairwise versions*.

Lemma 2. *Assume that G is a connected graph.*
(i) *Reallocation-proofness implies non-bossiness.*
(ii) *Reallocation-proofness is equivalent to the combination of pairwise reallocation-proofness and pairwise non-bossiness.*

Proof. To prove part (i), let f be a rule satisfying *reallocation-proofness*. Let $S \subseteq N$ be a connected coalition on G and $S \neq N$. Let $(c, E) \in \mathcal{D}$ and $c' \in \mathbb{R}_+^{N\times K}$ be such that $\bar{c}_S = \bar{c}'_S$ and $c_{N\backslash S} = c'_{N\backslash S}$. Let $x \equiv f(c, E)$ and $x' \equiv f(c', E)$. By *reallocation-proofness*, $\bar{x}_S = \bar{x}'_S$. Since G is a connected graph, there exists a node $i_1 \in N\backslash S$ that is adjacent to a node in S. Let $S_1 \equiv S \cup \{i_1\}$. Then S_1 is also connected and $c_{i_1} = c'_{i_1}$. Hence $\bar{c}_{S_1}(= \bar{c}_S + c_{i_1}) = \bar{c}'_{S_1}(= \bar{c}'_S + c'_{i_1})$ and so by *reallocation-proofness*, $\bar{x}_S + x_{i_1} = \bar{x}'_S + x'_{i_1}$. Since $\bar{x}_S = \bar{x}'_S$, $x_{i_1} = x'_{i_1}$. Suppose by induction that $k \leq |N\backslash S|$ and $i_1, \cdots, i_k \in N\backslash S$ are such that $S_k \equiv S \cup \{i_1, \cdots, i_k\}$ is connected, $\bar{c}_{S_k} = \bar{c}'_{S_k}$, and $x_{\{i_1,\cdots,i_k\}} = x'_{\{i_1,\cdots,i_k\}}$. If $N\backslash S_k = \emptyset$, we are done. If not, then since G is a connected graph, there exists a node $i_{k+1} \in N\backslash S_k$ that is adjacent to a node in S_k. Let $S_{k+1} \equiv S_k \cup \{i_{k+1}\}$. Then S_{k+1} is connected and since $\bar{c}_{S_k} = \bar{c}'_{S_k}$ and $c_{i_{k+1}} = c'_{i_{k+1}}$, $\bar{c}_{S_{k+1}} = \bar{c}'_{S_{k+1}}$. Hence by *reallocation-proofness*, $\bar{x}_{S_{k+1}} = \bar{x}'_{S_{k+1}}$. Since $\bar{x}_{S_k}(= \bar{x}_S + x_{i_1} + \cdots + x_{i_k}) = \bar{x}'_{S_k}(= \bar{x}'_S + x'_{i_1} + \cdots + x'_{i_k})$, $x_{k+1} = x'_{k+1}$. Therefore, $x_{\{i_1,\cdots,i_{k+1}\}} = x'_{\{i_1,\cdots,i_{k+1}\}}$. Since N is finite, the iteration will end after a finite number of steps and, at the end, we obtain $x_{N\backslash S} = x'_{N\backslash S}$.

By part (i), *reallocation-proofness* implies both *pairwise reallocation-proofness* and *pairwise non-bossiness*. To prove the converse, let f be a rule satisfying *pairwise reallocation-proofness* and *pairwise non-bossiness*. Let $S \subseteq N$ be connected. Let $(c, E), (c', E) \in \mathcal{D}$ be such that $\bar{c}_S = \bar{c}'_S$ and $c_{N\backslash S} = c'_{N\backslash S}$. We only have to show $\sum_{i\in S} f_i(c,E) = \sum_{i\in S} f_i(c',E)$ and $f_{N\backslash S}(c,E) = f_{N\backslash S}(c',E)$.

By Lemma 1, there exist a number r, $S_1, S_2, \cdots, S_r \in D_S$, and $c^1, c^2, \cdots, c^r \in \mathbb{R}_+^{N\times K}$ such that $\bar{c}^1_{S_1} = \bar{c}_{S_1}$, $c^1_{N\backslash S_1} = c_{N\backslash S_1}$, $c^r = c'$, and for each $m = 2, \cdots, r$, $\bar{c}^m_{S_m} = \bar{c}^{m-1}_{S_m}$ and $c^m_{N\backslash S_m} = c^{m-1}_{N\backslash S_m}$. By richness of $\mathcal{D}$, $(c^1, E), \cdots, (c^r, E) \in \mathcal{D}$. For each $m = 1, \cdots, r-1$, let $x^m \equiv f(c^m, E)$. Let $x \equiv f(c,E)$ and $x' \equiv f(c',E)$. Since $\bar{c}^1_{S_1} = \bar{c}_{S_1}$, then by *pairwise reallocation-proofness*, $\bar{x}^1_{S_1} = \bar{x}_{S_1}$. By *pairwise non-bossiness*, $x^1_{N\backslash S_1} = x_{N\backslash S_1}$. Since $S_1 \subseteq S$, then $\bar{x}^1_S = \bar{x}_S$ and $x^1_{N\backslash S} = x_{N\backslash S}$. For each $m = 2, \cdots, r$, since $\bar{c}^m_{S_m} = \bar{c}^{m-1}_{S_m}$, then by *pairwise reallocation-proofness*, $\bar{x}^m_{S_m} = \bar{x}^{m-1}_{S_m}$ and by *pairwise non-bossiness*, $x^m_{N\backslash S_m} = x^{m-1}_{N\backslash S_m}$. Since $S_m \subseteq S$, then $\bar{x}^m_S = \bar{x}^{m-1}_S$ and $x^m_{N\backslash S} = x^{m-1}_{N\backslash S}$. This shows $\bar{x}'_S = \bar{x}_S$ and $x'_{N\backslash S} = x_{N\backslash S}$. ∎

By Lemma 2, *reallocation-proofness* in all our results can be replaced with the combination of *pairwise reallocation-proofness* and *pairwise non-bossiness*.

3.1 Complete Graph

Reallocation-proofness under the unrestricted coalition structure (that is, when G is a complete graph) is studied by Ju, Miyagawa, and Sakai (2003). They offer the following characterization results.

Proposition 1 (Ju, Miyagawa, and Sakai 2003). *Assume that G is a complete graph and $|N| \geq 3$.*
(i) *A rule f on a rich domain $\mathcal{D}$ satisfies reallocation-proofness if and only if there exist two functions $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $\hat{W}\colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that, for each $(c,E) \in \mathcal{D}$ and each $i \in N$,*

$$f_i(c,E) = A_i(\bar{c},E) + \sum_{k\in K} \hat{W}_k(c_{ik}, \bar{c}, E),$$

and $\hat{W}(\cdot, \bar{c}, E)$ is additive.
(ii) *A rule on a rich domain $\mathcal{D}$ satisfies reallocation-proofness and one-sided boundedness if and only if it is a generalized proportional rule.*
(iii) *A rule on a rich domain $\mathcal{D}$ satisfies pairwise reallocation-proofness, efficiency, no award for null, and non-negativity (or no transfer paradox) if and only if it is a proportional rule.*

By virtue of part (i) of Proposition 1, within *reallocation-proof* rules, necessary and sufficient conditions for other axioms can be stated as conditions for the two functions $A(\cdot)$ and $\hat{W}(\cdot)$ as follows:

Proposition 2 (Ju, Miyagawa, and Sakai 2003). *Assume that G is a complete graph. Let f be a reallocation-proof rule represented by $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $\hat{W}\colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ as in part* (i) *of Proposition 1. Then f satisfies*
(i) *Efficiency if and only if for each $(c, E) \in \mathcal{D}$,*

$$\sum_{i \in N} A_i(\bar{c}, E) + \sum_{k \in K} \hat{W}_k(\bar{c}_k, \bar{c}, E) = E.$$

(ii) *No award for nulls if and only if for each $(c, E) \in \mathcal{D}$ and each $i \in N$,*

$$A_i(\bar{c}, E) = 0.$$

(iii) *Non-negativity if and only if f satisfies one-sided boundedness and, for each $(c, E) \in \mathcal{D}$,*

$$A_i(\bar{c}, E) \geq 0 \text{ for each } i \in N,$$
$$\min_{j \in N} A_j(\bar{c}, E) + \sum_{k \in K} \min\{0, \hat{W}_k(\bar{c}_k, \bar{c}, E)\}E \geq 0.$$

(iv) *No transfer paradox if and only if for each $(c, E) \in \mathcal{D}$ and each $k \in K$, $\hat{W}_k(\cdot, \bar{c}, E)$ is non-decreasing.*

In the next subsections, we will consider three types of "incomplete" graphs which are crucial for establishing our theorem.

3.2 Tree

In this section, we consider the case when G is a tree.

The next result is a characterization of *reallocation-proof* rules when G is a tree.

Proposition 3. *Assume that G is a tree. Then a rule f on a rich domain $\mathcal{D}$ satisfies reallocation-proofness if and only if f is represented by a function $H\colon \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ such that for each $(c, E) \in \mathcal{D}$ and each $i \in N$,*

$$f_i(c, E) = \begin{cases} H_i\left(\bar{c}_{s(i)}, \bar{c}, E\right) - \sum_{j \in sm(i)} H_j\left(\bar{c}_{s(j)}, \bar{c}, E\right), & \text{if } sm(i) \neq \emptyset, \\ H_i(c_i, \bar{c}, E), & \text{if } sm(i) = \emptyset, \end{cases} \tag{4}$$

where $s(\cdot)$ and $sm(\cdot)$ are defined on a directed tree $G(i^*)$ with root $i^* \in N$.

Proof. Let $G \equiv (N, D)$ be a tree. Fix $i^* \in N$ and consider the directed tree with root i^*, $G(i^*)$. Let f be a rule given by (4). By Lemma 2, to prove *reallocation-proofness*, we only have to prove *pairwise reallocation-proofness* and *pairwise* *non-bossiness*. Note that we can rewrite (4) equivalently as follows: for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = \begin{cases} H_i\left(\bar{c}_{s(i)}, \bar{c}, E\right) - \sum_{j \in s^0(i)} f_j(c, E), & \text{if } s^0(i) \neq \emptyset, \\ H_i(c_i, \bar{c}, E), & \text{if } s^0(i) = \emptyset. \end{cases} \qquad (\star)$$

Thus, for each $(c, E) \in \mathcal{D}$ and each $i \in N$, $H_i\left(\bar{c}_{s(i)}, \bar{c}, E\right)$ is the total award for agent i and i's successors, that is,

$$H_i\left(\bar{c}_{s(i)}, \bar{c}, E\right) = \sum_{j \in s(i)} f_j(c, E). \qquad (\star\star)$$

Note also that for each $i \in N$ and each $j \in sm(i)$,

$$\begin{aligned} f_i(c, E) + f_j(c, E) = & \left[H_i\left(\bar{c}_{s(i)}, \bar{c}, E\right) - H_j\left(\bar{c}_{s(j)}, \bar{c}, E\right) - \sum_{j' \in sm(i) \setminus \{j\}} H_{j'}\left(\bar{c}_{s(j')}, \bar{c}, E\right) \right] \\ & + \left[H_j\left(\bar{c}_{s(j)}, \bar{c}, E\right) - \sum_{h \in sm(j)} H_h\left(\bar{c}_{s(h)}, \bar{c}, E\right) \right]. \end{aligned}$$

Thus

$$\begin{aligned} f_i(c, E) + f_j(c, E) &= H_i\left(\bar{c}_{s(i)}, \bar{c}, E\right) - \sum_{j' \in sm(i) \setminus \{j\}} H_{j'}\left(\bar{c}_{s(j')}, \bar{c}, E\right) \\ &\quad - \sum_{h \in sm(j)} H_h\left(\bar{c}_{s(h)}, \bar{c}, E\right). \end{aligned} \qquad (\dagger)$$

The first term in $(\dagger)$ depends on (c_i, c_j) only through $c_i + c_j$ and $\bar{c}$, and the remaining two terms depend on (c_i, c_j) only through $\bar{c}$. Therefore, the coalition of i and j cannot change their total award by changing the sum of their characteristic vectors.

To prove *pairwise non-bossiness*, let $h \in s(i) \setminus \{i, j\}$. If h is an end node, $f_h(c, E) = H_h(c_h, \bar{c}, E)$. Then $f_h(c, E)$ depends on (c_i, c_j) only through $\bar{c}$. Hence $f_h(c, E)$ is not affected by any reallocation of characteristic vectors of i and j. Now moving backward and using induction argument and $(\star)$, we can show that $f_h(c, E)$ is not affected by any reallocation of characteristic vectors of i and j. The same argument will show that for each h who is neither a successor nor a predecessor of i, h's award is not affected by any reallocation of characteristic vectors among i and j. Now consider h who is a predecessor of i. Then $i, j \in s^0(h)$. Assume that h is an immediate predecessor of i. By $(\star)$, h's award depends on (c_i, c_j) only through $\bar{c}_{s(h)}$, $\bar{c}$, and $\sum_{k \in s^0(h)} f_k(c, E)$. By the previous argument, none of these factors is affected by a reallocation of characteristic vectors among i and j. Therefore, h's award is not affected either. Arguing inductively, we obtain the same conclusion for each predecessor of i. Therefore f satisfies *pairwise non-bossiness*.

To prove the converse, let f be a rule satisfying *reallocation-proofness*. Then by Lemma 2, it also satisfies *non-bossiness*. For each $i \in N$, define H as follows: for each $i \in N$ and each $(x, y, E) \in \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++}$ with $x \leq y$,

$$H_i(x, y, E) \equiv \sum_{j \in s(i)} f_j(c, E),$$

for some $(c, E) \in \mathcal{D}$ with $\bar{c}_{s(i)} = x$ and $\bar{c} = y$. For all other $(x, y, E) \in \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++}$, set $H_i(x, y, E)$ arbitrarily. Then $(\star)$ follows directly from the definition of H and we obtain (4). Therefore, we only have to show that H is well-defined. Let $c, c' \in \mathbb{R}_+^{N \times K}$ be such that $\bar{c}_{s(i)} = \bar{c}'_{s(i)} = x$ and $\bar{c} = \bar{c}' = y$. Let $x \equiv f(c, E)$, $x' \equiv f(c', E)$, and $x'' \equiv f(c_{s(i)}, c'_{N \setminus s(i)}, E)$. Since $N \setminus s(i)$ is connected, then by *reallocation-proofness* and *non-bossiness*, $x_{s(i)} = x''_{s(i)}$ (and $\bar{x}_{N \setminus s(i)} = \bar{x}''_{N \setminus s(i)}$). Since $s(i)$ is also connected, then by *reallocation-proofness* and *non-bossiness*, $\bar{x}''_{s(i)} = \bar{x}'_{s(i)}$ (and $x''_{N \setminus s(i)} = x'_{N \setminus s(i)}$). Therefore, $\bar{x}_{s(i)} = \bar{x}'_{s(i)}$, which is what we need to show for the well-definedness of H. ∎

Although the domain of H_i is stated as $\mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++}$ in Proposition 3, only its subset $\{(x, y, E) \in \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++}$: for some $(c, E) \in \mathcal{D}$, $\bar{c}_{s(i)} = x$ and $\bar{c} = y\}$ matters. What values H_i takes outside this subset is not relevant to our result and in (4). In what follows we will say that H or H_i has a certain property, when it has the property only over this subset.

Note that when f is a generalized proportional rule associated with (A, W), then for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$H_i(c, E) = \sum_{j \in s(i)} A_j(\bar{c}, E) + \sum_{k \in K} \frac{\bar{c}_{s(i)k}}{\bar{c}_k} W_k(\bar{c}, E) E.$$

Proposition 4. *Assume that G is a tree. Let f be a* reallocation-proof *rule represented by $H\colon \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^N$ as in Proposition 3, where $s(\cdot)$ and $sm(\cdot)$ be defined on the directed tree $G(i^*)$ with root $i^* \in N$. Then f satisfies*
1. *Efficiency if and only if $H_{i^*}(\bar{c}, \bar{c}, E) = E$ for each $(c, E) \in \mathcal{D}$.*
2.1. *Assume that $G(i^*)$ has a node $i \neq i^*$ with at least two immediate successors (that is, G is a non-linear tree). Then f satisfies* no award for null *if and only if $H_1 = \cdots = H_N \equiv H^0$ and for each $(c, E) \in \mathcal{D}$, $H_0(\cdot, \bar{c}, E)$ is additive.*

Hence, for each $(c, E) \in \mathcal{D}$*,* $H_0(0, \bar{c}, E) = 0$ *and* $H_0(\cdot, \bar{c}, E)$ *can be decomposed into* K *functions as follows*

$$\begin{aligned} f_i(c, E) &= H_0(c_i, \bar{c}, E) \\ &= \sum_{k \in K} \hat{W}_k(c_{ik}, \bar{c}, E), \end{aligned}$$

where $\hat{W}_k(c_{ik}, \bar{c}, E) \equiv H_0(c_{ik}\mathbf{u}_k, \bar{c}, E)$*, denoting the* k^{th} *unit vector of* $\mathbb{R}^K$ *by* $\mathbf{u}_k$*, and so* $\hat{W}_k(\cdot, \bar{c}, E)$ *is additive.*
2.2. *When* G *is a line,* f *satisfies* no award for null *if and only if for each* $(\bar{c}, E) \in \mathbb{R}^K_{++} \times \mathbb{R}_{++}$

$$H_1 = H_2 = \cdots = H_N \equiv H_0,$$
$$H_0(0, \bar{c}, E) = 0.$$

3. Non-negativity *if and only if for each* $i \in N$*, each* $x, y \in \mathbb{R}^K_+$*, each* $E \in \mathbb{R}_{++}$*, and each* $(a_j)_{j \in sm(i)} \in \mathbb{R}^{sm(i) \times K}_+$ *with* $0 \leq \sum_{j \in sm(i)} a_j \leq x \leq y$*.*

$$\begin{aligned} H_i(x, y, E) &\geq \sum_{j \in sm(i)} H_j(a_j, y, E), \text{ if } sm(i) \neq \emptyset, \\ H_i(x, y, E) &\geq 0, \text{ if } sm(i) = \emptyset. \end{aligned}$$

4. No transfer paradox *if and only if* $H_i(\cdot, \bar{c}, E)$ *is non-decreasing for each* $i \in N$ *and each* $(c, E) \in \mathcal{D}$*.*
Thus, if f *satisfies* no award for null*, then* non-negativity *is equivalent to* no transfer paradox*.*

Proof. **1.** This follows from $s(i^*) = N$ and the fact that for each (c, E) and each $i \in N$, $H_i(\bar{c}_{s(i)}, \bar{c}, E) = \sum_{j \in s(i)} f_j(c, E)$.

2. Let f satisfy *no award for null*. Then for each $(c, E) \in \mathcal{D}$ with $c_i = 0$, $f_i(c, E) = 0$. Then by (4), for each $(c, E) \in \mathcal{D}$ with $c_i = 0$,

$$H_i\left(\sum_{j \in sm(i)} \bar{c}_{s(j)}, \bar{c}, E\right) = \sum_{j \in sm(i)} H_j(\bar{c}_{s(j)}, \bar{c}, E). \tag{5}$$

Since this holds for each $(c, E) \in \mathcal{D}$ with $c_i = 0$, then we obtain: for each $i \in N$, each $(x_j)_{j \in sm(i)} \in \mathbb{R}^{sm(i) \times K}_+$, and each $(y, E) \in \mathbb{R}^K_{++} \times \mathbb{R}_{++}$, if there is $(c, E) \in \mathcal{D}$ such that $c_i = 0$, $\bar{c}_{s(j)} = x_j$ for each $j \in sm(i)$, and $\bar{c} = y$, then

$$H_i(\bar{x}_{sm(i)}, y, E) = \sum_{j \in sm(i)} H_j(x_j, y, E). \tag{$*$}$$

By *no award for null* and ($\star\star$) in the proof of Proposition 3, for each $i \in N$ and each $(c, E) \in \mathcal{D}$, if all successors of i have the zero characteristic vector, then they all receive nothing and so $\sum_{j \in s(i)} f_j(c, E) = 0$. Hence, for each $(y, E) \in \mathbb{R}^K_{++} \times \mathbb{R}^K_{++}$,

$$H_i(0, y, E) = 0. \tag{$**$}$$

Let $i \in N$ and $j \in sm(i)$. Let $(c, E) \in \mathcal{D}$ be such that $c_i = 0$ and for each $h \in s(i) \backslash \{j\}$, $c_h = 0$. Then by (5) and ($**$), $H_i(c_j, \bar{c}, E) = H_j(c_j, \bar{c}, E)$. Since this holds for each c_j with $0 \leq c_j \leq \bar{c}$, $H_i = H_j$. Using this and the tree structure of G, we show $H_1 = \cdots = H^N$. Let H_0 be the common function. For each $(c, E) \in \mathcal{D}$, if there is a node $i \in N \backslash \{i^*\}$ with at least two immediate successors, we obtain additivity of $H_0(\cdot, \bar{c}, E)$ from ($*$) (note that if ($*$) holds for $i = i^*$, then we can only obtain the limited additivity of $H_0(\cdot, \bar{c}, E)$ saying that for each $x, x' \in \mathbb{R}^K_+$, if $x + x' = \bar{c}$, $H_0(x, \bar{c}, E) + H_0(x', \bar{c}, E) = H_0(x + x', \bar{c}, E)$). Using additivity of $H_0(\cdot, \bar{c}, E)$ and (4) in Proposition 3, we show $f_i(c, E) = H_0(c_i, \bar{c}, E)$.

The converse follows easily from the fact that $H_0(0, \bar{c}, E) = 0$ and $f_i(c, E) = H_0(c_i, \bar{c}, E)$ for each $(c, E) \in \mathcal{D}$.

3. This part follows directly from (4).

4. Assume that f satisfies *no transfer paradox*. Let i be a terminal node, that is, $s^0(i) = \emptyset$. Then since $f_i(c, E) = H_i(c_i, \bar{c}, E)$, $H_i(\cdot, \bar{c}, E)$ is non-decreasing, for each $(c, E) \in \mathcal{D}$. Let j be such that for each $i \in s^0(j)$, $s^0(i) = \emptyset$. Then $f_j(c, E) = H_j\left(\bar{c}_{s(j)}, \bar{c}, E\right) - \sum_{i \in sm(j)} H_i(c_i, \bar{c}, E)$ and for each $i \in sm(j)$, $H_i(\cdot, \bar{c}, E)$ is non-decreasing. Consider transferring $t \in [0, c_i]$ from $h \in p^0(j)$ to j. Then by *no transfer paradox*, j's award must not decrease. Thus $H_j\left(\bar{c}_{s(j)} + t, \bar{c}, E\right) - \sum_{i \in sm(j)} H_i(c_i, \bar{c}, E) \geq H_j\left(\bar{c}_{s(j)}, \bar{c}, E\right) - \sum_{i \in sm(j)} H_i(c_i, \bar{c}, E)$. Hence, for each $(c, E) \in \mathcal{D}$ and each $t \in [0, c_i]$, $H_j\left(\bar{c}_{s(j)} + t, \bar{c}, E\right) \geq H_j\left(\bar{c}_{s(j)}, \bar{c}, E\right)$. This shows that $H_j(\cdot, \bar{c}, E)$ is non-decreasing. Proceeding backward, we complete our proof. The converse is shown easily. ∎

Note that when G is a non-linear tree, adding *no award for null*, we obtain a subfamily of rules that are characterized in part (i) of Proposition 1 and that have $A_i(\cdot) = 0$ for each $i \in N$. Thus, given *no award for null*, *reallocation-proofness* on a tree is equivalent to *reallocation-proofness* on a complete graph. Therefore, all earlier characterization results based on *reallocation-proofness* on a complete graph and *no award for null* continue to hold on a tree.

Line

If G is a line, then given an end node $i^* \in N$ and the directed line $G(i^*)$, any node can have at most one immediate successor. Thus, the immediate successor of i, if any, can be denoted by $sm(i)$ with a slight abuse of notation (recall that $sm(i)$ is the set of immediate successors). Then from Proposition 3, we obtain:

Corollary 1. *Assume that G is a line. A rule f on a rich domain $\mathcal{D}$ satisfies reallocation-proofness if and only if f is represented by a function $H\colon \mathbb{R}_{+}^{K} \times \mathbb{R}_{++}^{K} \times \mathbb{R}_{++} \to \mathbb{R}^{N}$ such that for each $(c, E) \in \mathcal{D}$ and each $i \in N$,*

$$f_i(c,E) = \begin{cases} H_i\left(\bar{c}_{s(i)}, \bar{c}, E\right), & \text{if } sm(i) = \emptyset; \\ H_i\left(\bar{c}_{s(i)}, \bar{c}, E\right) - H_{sm(i)}\left(\bar{c}_{s(sm(i))}, \bar{c}, E\right), & \text{if } sm(i) \neq \emptyset, \end{cases} \tag{6}$$

where, for an end node $i^* \in N$, $s(\cdot)$ and $sm(\cdot)$ are defined on the directed line $G(i^*)$.

Combining *reallocation-proofness*, *efficiency*, and *no award for null*, we obtain:

Corollary 2. *Assume that G is a line. A rule f on a rich domain $\mathcal{D}$ satisfies reallocation-proofness, efficiency, and no award for null if and only if f is represented by a function $H_0\colon \mathbb{R}_{+}^{K} \times \mathbb{R}_{++}^{K} \times \mathbb{R}_{++} \to \mathbb{R}$ such that for each $(c, E) \in \mathcal{D}$ and each $i \in N$, $H_0(0, \bar{c}, E) = 0$, $H_0(\bar{c}, \bar{c}, E) = E$, and*

$$f_i(c,E) = \begin{cases} H_0(c_i, \bar{c}, E), & \text{if } sm(i) = \emptyset, \\ H_0\left(\bar{c}_{s(i)}, \bar{c}, E\right) - H_0\left(\bar{c}_{s(sm(i))}, \bar{c}, E\right), & \text{if } sm(i) \neq \emptyset, \end{cases}$$

where, for an end node $i^* \in N$, $s(\cdot)$ and $sm(\cdot)$ are defined on the directed line $G(i^*)$.

Proposition 4 (parts 2.1 and 2.2) shows that when *no award for null* is imposed, there is a remarkable difference between the linear tree case and the non-linear tree case. As shown in Corollary 2, in the case of linear tree, we have "hierarchical division rules" satisfying *reallocation-proofness* and *no award for null*. These rules are not necessarily a member of the family of rules characterized in Proposition 1. Among these rules, only those rules characterized in Proposition 1 are left when G is a non-linear tree.

3.3 Rigid* Graph

In this section, we consider the case when G is rigid*.

In the next lemma, we show that when G is rigid*, *reallocation-proofness* under $\mathcal{C}(G)$ is equivalent to *reallocation-proofness* under the unrestricted coalition structure.

Lemma 3. *Given a connected graph $G \equiv (N, D)$, let f be a rule satisfying* reallocation-proofness*. For each $T \subseteq N$, if no node in $N \backslash T$ is a connection node, then for each $(c, E), (c', E) \in \mathcal{D}$ with $\bar{c}_T = \bar{c}'_T$ and $c_{N \backslash T} = c'_{N \backslash T}$,*

$$\begin{aligned} \sum_{i \in T} f_i(c, E) &= \sum_{i \in T} f_i(c', E), \\ f_{N \backslash T}(c, E) &= f_{N \backslash T}(c', E). \end{aligned}$$

Therefore, if G is rigid, then* reallocation-proofness *under $\mathcal{C}(G)$ is equivalent to* reallocation-proofness *under the unrestricted coalition structure.*

Proof. Let $G \equiv (N, D)$ be a connected graph. Let f be a rule satisfying *reallocation-proofness* under $\mathcal{C}(G)$. Then by Lemma 2, f satisfies *non-bossiness*. Let $T \subseteq N$. Assume that no node in $N \backslash T$ is a connection node. Let $(c, E), (c', E) \in \mathcal{D}$ be such that $\bar{c}_T = \bar{c}'_T$ and $c_{N \backslash T} = c'_{N \backslash T}$. Let $x \equiv f(c, E)$ and $x' \equiv f(c', E)$. We only have to show $\bar{x}_T = \bar{x}'_T$ and $x_{N \backslash T} = x'_{N \backslash T}$. Since N is connected, by *reallocation-proofness*,

$$\bar{x}_N = \bar{x}'_N. \tag{7}$$

For each $i \in N \backslash T$, since i is not a connection node, $N \backslash \{i\}$ is connected. Since $\bar{c}_{N \backslash \{i\}} = \bar{c}'_{N \backslash \{i\}}$, then by *reallocation-proofness* and *non-bossiness*, $x_i = x'_i$. Hence $x_{N \backslash T} = x'_{N \backslash T}$. Combining this with (7), we obtain $\bar{x}_T = \bar{x}'_T$. ∎

We will show later that rigidity* of G is the necessary and sufficient condition for the equivalence between *reallocation-proofness* under $\mathcal{C}(G)$ and *reallocation-proofness* under the unrestricted coalition structure.

It follows from this lemma and Proposition 1 that:

Proposition 5. *Assume that graph $G \equiv (N, D)$ is rigid* and $|N| \geq 3$.*
(i) *A rule f on a rich domain $\mathcal{D}$ satisfies* reallocation-proofness *if and only if f is represented by two functions $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $\hat{W}\colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that for each $(c, E) \in \mathcal{D}$ and each $i \in N$,*

$$f_i(c, E) = A_i(\bar{c}, E) + \sum_{k \in K} \hat{W}_k(c_{ik}, \bar{c}, E),$$

and $\hat{W}(\cdot, \bar{c}, E)$ is additive.
(ii) *A rule on a rich domain satisfies* reallocation-proofness *and* one-sided boundedness *if and only if it is a generalized proportional rule.*
(iii) *A rule on a rich domain satisfies* reallocation-proofness, no award for null, *and* non-negativity *(or* no transfer paradox*) if and only if it is a proportional rule.*

3.4 Rigid Graph

In this section, we consider the case when G is rigid.

In the next lemma, we show that each rigid graph is composed of maximal rigid* subgraphs connected with each other by connection nodes.

Lemma 4. *Assume that $G \equiv (N, D)$ is a rigid graph.*
(i) *The set of nodes N is uniquely divided into a finite number of subsets $N_1, \cdots, N_L$ with $\cup_{l=1}^{L} N_l = N$ such that for each $l = 1, \cdots, L$, $|N_l| \geq 3$ and G_{N_l} is a maximal rigid* subgraph on G.*
(ii) *There is no cycle of successively intersecting sets among $N_1, \cdots, N_L$, that is, there is no $r \geq 3$ and no $N_{l_1}, \cdots, N_{l_r} \in \{N_1, \cdots, N_L\}$ such that $N_{l_1} \cap N_{l_2} \neq \emptyset, \cdots, N_{l_{r-1}} \cap N_{l_r} \neq \emptyset$, and $N_{l_1} = N_{l_r}$.*

The proof is in Appendix A.

By Lemma 4, N has the unique family of subsets $N_1, \cdots, N_L$ such that for each $l \in \{1, \cdots, L\}$, $|N_l| \geq 3$ and G_{N_l} is a maximal rigid* subgraph. In this case, we say that *rigid graph G is composed of maximal rigid* subgraphs $G_{N_1}, \cdots, G_{N_L}$*. Let $\mathcal{N}^*(G) \equiv \{N_1, \cdots, N_L\}$ and $\mathcal{R}^*(G) \equiv \{G_{N_1}, \cdots, G_{N_L}\}$. For each $l \in \{1, \cdots, L\}$, let

$$C(N_l) \equiv \{i \in N_l : i \text{ is a connection node on } G\}$$

be the set of connection nodes in N_l on graph G. For each $l \in \{1, \cdots, L\}$ and each $i \in N_l$, let

$$S(i, N_l) \equiv \{j \in N \backslash [N_l \backslash \{i\}] : i \text{ is between } j \text{ and any node in } N_l\}$$

be the set of nodes outside $N_l \backslash \{i\}$ that can be connected with any node in N_l only through i. Note $i \in S(i, N_l)$. Note also that $S(i, N_l)$ is not a singleton if and only if $i \in C(N_l)$. For example, if G is composed of two rigid* subgraphs G_{N_1} and G_{N_2} and the connection node is $\hat{\imath}$, then $S(\hat{\imath}, N_1) = N_2$, $S(\hat{\imath}, N_2) = N_1$, and $C(N_1) = C(N_2) = \{\hat{\imath}\}$.

Proposition 6. *Assume that a rigid graph $G \equiv (N, D)$ with $|N| \geq 3$ is composed of L maximal rigid* subgraphs $G_{N_1}, \cdots, G_{N_L}$: that is, $\mathcal{R}^*(G) \equiv \{G_{N_1}, \cdots, G_{N_L}\}$. Then a rule over a rich domain $\mathcal{D}$ satisfies* reallocation-proofness *if and only if there exists a list of functions $(A^l : \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{N_l}, \hat{W}^l : \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K)_{l \in \{1, \cdots, L\}}$ such that for each $(c, E) \in \mathcal{D}$ and each $l \in \{1, \cdots, L\}$ and each $i \in N_l$, if $i \in N_l \backslash C(N_l)$,*

$$f_i(c, E) = A^l_i(\bar{c}, E) + \sum_{k \in K} \hat{W}^l_k(c_{ik}, \bar{c}, E); \tag{8}$$

and if $i \in C(N_l)$ and $\{G_{N_{l_1}}, \cdots, G_{N_{l_r}}\}$ is the set of all rigid subgraphs in $\mathcal{R}^*(G)$ other than G_{N_l}, to which i also belongs,*

$$\begin{aligned} f_i(c,E) &= A_i^l(\bar{c},E) + \sum_{k\in K} \hat{W}_k^l\left(\bar{c}_{S(i,N_l)k}, \bar{c}, E\right) \\ &\quad - \sum_{s=1}^{r} \sum_{j\in N_{l_s}\setminus\{i\}} A_j^{l_s}(\bar{c},E) \\ &\quad - \sum_{s=1}^{r} \sum_{k\in K} \hat{W}_k^{l_s}\left(\bar{c}_{[N_{l_s}\setminus C(N_{l_s})]k} + \sum_{j\in C(N_{l_s})\setminus\{i\}} \bar{c}_{S(j,N_{l_s})k}, \bar{c}, E\right), \end{aligned} \tag{9}$$

where for each $l, l' \in \{1, \cdots, L\}$, $\hat{W}^l(\cdot, \bar{c}, E)$ is additive and

$$\sum_{i\in N_l} A_i^l(\bar{c},E) + \sum_{k\in K} \hat{W}_k^l(\bar{c}_k, \bar{c}, E) = \sum_{i\in N_{l'}} A_i^{l'}(\bar{c},E) + \sum_{k\in K} \hat{W}_k^{l'}(\bar{c}_k, \bar{c}, E). \tag{10}$$

The proof is in Appendix B.

Remark 1. Note that when G is a rigid* graph, $L = 1$ and Proposition 6 reduces to part (i) of Proposition 5.

4 Theorem

We now consider the most general case when G is a connected graph.

The next lemma says that every connected graph is uniquely decomposed into a family of maximal rigid subgraphs.

Lemma 5. *Assume that $G \equiv (N, D)$ is a connected graph.*
(i) *The set of nodes N is uniquely partitioned into a finite number of subsets $N_1, \cdots, N_L$ such that for each $l = 1, \cdots, L$, $|N_l| = 1$ or $|N_l| \geq 3$ and G_{N_l} is a maximal rigid subgraph on G.*
(ii) *There is no cycle of sets among $N_1, \cdots, N_L$, which are successively connected by connection edges; that is, there is no $r \geq 3$ and no $N_{l_1}, \cdots, N_{l_r} \in \{N_1, \cdots, N_L\}$ such that $N_{l_1} = N_{l_r}$ and for two sequences of nodes, $i_1 \in N_{l_1}, \cdots, i_{r-1} \in N_{r-1}$ and $j_2 \in N_{l_2}, \cdots, j_r \in N_r$, we have $i_1j_2, i_2j_3, \cdots, i_{r-1}j_r \in D$.*

The proof is in Appendix A.

By Lemma 5, N is partitioned into maximal rigid subgraphs and these subgraphs are located with a tree structure. Formally:

Definition 4 (Tree of Maximal Rigid Subgraphs). Given a connected graph $G \equiv (N, D)$, let N be partitioned into $N_1, \cdots, N_L$ such that for each $l = 1, \cdots, L$, G_{N_l} is a maximal rigid subgraph. We now define a graph $\mathcal{G}$ of which nodes are composed of these subgraphs. Formally, let $\mathcal{N} \equiv \{N_1, \cdots, N_L\}$ be the set of nodes. For each $l, l' \in \{1, \cdots, L\}$, $\{N_l, N_{l'}\}$ is an edge of $\mathcal{G}$ if there is an edge of the original graph G, which connects N_l and $N_{l'}$, that is, for some $i \in N_l$ and $i' \in N_{l'}$, $ii' \in D$. Denote the set of edges of $\mathcal{G}$ by $\mathcal{E}$. Then $\mathcal{G} \equiv (\mathcal{N}, \mathcal{E})$ is a tree because of part (ii) of Lemma 5.

Let $\mathcal{R} \equiv \{G_{N_1}, \cdots, G_{N_L}\}$ be the set of maximal rigid subgraphs on G. Note that for each $l = 1, \cdots, L$, $|N_l| = 1$ or $|N_l| \geq 3$. By Lemma 4, for each $l = 1, \cdots, L$, N_l is again divided into a finite number $L_l \in \mathbb{N}$ of subsets, denoted by $N_{l1}, \cdots, N_{lL_l}$, such that for each $m = 1, \cdots, L_l$, $G_{N_{lm}}$ is a maximal rigid* subgraph on G_{N_l}.

Next we define a family of rules that are similar to rules characterized in Proposition 6 on each rigid subgraph $G_{N_l} \in \mathcal{R}$ with $|N_l| \geq 3$ and that are similar to rules characterized in Proposition 3 when we view the total award of each group $N_l \in \mathcal{N}$ as the award for node N_l on the tree $\mathcal{G}$.

Pick an arbitrary $N_{l^*} \in \mathcal{N}$ and let $\mathcal{G}(N_{l^*})$ be the directed tree with root N_{l^*}. Then we can define successors and predecessors on $\mathcal{G}(N_{l^*})$. We use the same notation as in Section 3.2 for the set of successors $s(\cdot)$, the set of immediate successors $sm(\cdot)$, the set of predecessors $p(\cdot)$, and immediate predecessor $pm(\cdot)$. But note that variables of these functions are different because now the set of nodes is $\mathcal{N}$, while it is N in Section 3.2. We also use notation $s^0(\cdot)$ and $p^0(\cdot)$ as used in Section 3.2. For each $l \in \{1, \cdots, L\}$, let

$$\cup s(N_l) \equiv \bigcup_{N_{l'} \in s(N_l)} N_{l'}$$

be the union of all $N_{l'} \in \mathcal{N}$ that succeeds N_l on $\mathcal{G}(N_{l^*})$. Similarly, let

$$\cup s^o(N_l) \equiv \bigcup_{N_{l'} \in s^o(N_l)} N_{l'}.$$

For each $l \in \{1, \cdots, L\}$ and each $m \in \{1, \cdots, L_l\}$, let

$$\begin{aligned} C(N_l) &\equiv \{j \in N_l : j \text{ is a connection node on } G\}; \\ C(N_{lm}, G_{N_l}) &\equiv \{j \in N_{lm} : j \text{ is a connection node on } G_{N_l}\}. \end{aligned}$$

Let

$$C^*(N_l) \equiv \{i \in C(N_l) : \text{for some } N_{l'} \in sm(N_l) \text{ and some } j \in N_l,\ ij \in D\}$$

be the set of all connection nodes $i \in N_l$ on G, which belongs to a connection edge connecting N_l to an immediate successor of N_l on $\mathcal{G}$. For each $l \in \{1, \cdots, L\}$ and each $i \in N_l$, let

$$\begin{aligned} sm\left(N_l; i\right) &\equiv \{N_k \in s^0\left(N_l\right) : \text{for some } j \in N_k,\, ij \in D\}, \\ s^0\left(N_l; i\right) &\equiv \cup_{N_k \in sm(N_l;i)} s\left(N_k\right). \end{aligned}$$

That is, $sm(N_l; i)$ is the set of immediate successors of N_l, connected with N_l through i, and $s^0(N_l; i)$ is the set of all successors of N_l that succeeds i. Note that if $i \notin C^*(N_l)$, $sm(N_l; i) = s^0\left(N_l; i\right) = \emptyset$. Following the previous notational convention, let

$$\cup s^0\left(N_l; i\right) \equiv \cup_{N_k \in sm(N_l;i)} \cup_{N_{k'} \in s(N_k)} N_{k'}.$$

For each $i \in N_{lm}$, let

$$S\left(i, N_{lm}\right) \equiv \{j \in N_l \backslash \left[N_{lm} \backslash \{i\}\right] : i \text{ is between } j \text{ and each node in } N_{lm} \text{ on } G_{N_l}\}.$$

It should be noted that $S\left(i, N_{lm}\right)$ is defined on the subgraph G_{N_l} and $i \in S\left(i, N_{lm}\right)$, and that $S\left(i, N_{lm}\right)$ is not a singleton if and only if $i \in C\left(N_{lm}, G_{N_l}\right)$.

Let $H\colon \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L$ be a function such that for each $l \in \{1, \cdots, L\}$, H_l describes the total award of all agents in $\cup s\left(N_l\right)$, as a function of the sum of characteristic vectors of these agents, $\bar{c}$, and E. We now define the family of rules represented by such a function H and a list of functions

$$\left(\left(A^{lm}\colon \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{N_{lm}}, \hat{W}^{lm}\colon \mathbb{R}_+ \times \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K\right)_{m=1}^{L_l}\right)_{l=1}^{L},$$

where for each $l \in \{1, \cdots, L\}$, each $m \in \{1, \cdots, L_l\}$, and each $(c, E) \in \mathcal{D}$, $\hat{W}^{lm}\left(\cdot, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right)$ is additive.

Definition 5 (HAW-Family). A rule f is in the *HAW-Family* if f is represented by a list of functions,

$$\begin{aligned} H\colon \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} &\to \mathbb{R}^L; \\ ((A^{lm}\colon \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} &\to \mathbb{R}^{N_{lm}}, \hat{W}^{lm}\colon \mathbb{R}_+ \times \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K)_{m=1}^{L_l})_{l=1}^{L} \end{aligned}$$

as follows: for each $(c, E) \in \mathcal{D}$, each $l \in \{1, \cdots, L\}$, each $m \in \{1, \cdots, L_l\}$, and each $i \in N_{lm}$,
(i) if $i \in N_l \backslash (C^*\left(N_l\right) \cup C\left(N_l, G_{N_l}\right))$,

$$f_i\left(c, E\right) = A^{lm}_i\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}^{lm}_k(c_{ik}, \bar{c}_{\cup s(N_l)}, \bar{c}, E), \tag{11}$$

(ii) if $i \in C(N_l, G_{N_l}) \backslash C^*(N_l)$ and $\{G_{N_{lm_1}}, \cdots, G_{N_{lm_r}}\}$ is the set of all rigid* subgraphs among $\{G_{N_{l1}}, \cdots, G_{N_{lL_l}}\} \backslash \{G_{N_{lm}}\}$, to which i also belongs,

$$
\begin{aligned}
f_i(c, E) = {} & A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}\left(\bar{c}_{S(i, N_{lm})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\
& - \sum_{s=1}^{r} \sum_{j \in N_{lm_s} \backslash \{i\}} A_j^{lm_s}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\
& - \sum_{s=1}^{r} \sum_{k \in K} \hat{W}_k^{lm_s}\left(\bar{c}_{\left[N_{lm_s} \backslash C(N_{lm_s}, G_{N_l})\right]k} + \sum_{j \in C(N_{lm_s}, G_{N_l}) \backslash \{i\}} \bar{c}_{S(j, N_{lm_s})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right),
\end{aligned}
\tag{12}
$$

(iii) if $i \in C^*(N_l) \cap (N_l \backslash C(N_l, G_{N_l}))$,

$$
\begin{aligned}
f_i(c, E) &= A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}(c_{ik}, \bar{c}_{\cup s(N_l)}, \bar{c}, E) \\
&\quad - \sum_{l' : N_{l'} \in sm(N_l; i)} H_{l'}\left(\bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right),
\end{aligned}
\tag{13}
$$

(iv) if $i \in C^*(N_l) \cap C(N_l, G_{N_l})$,

$$
\begin{aligned}
f_i(c, E) = {} & A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}\left(\bar{c}_{S(i, N_{lm})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\
& - \sum_{s=1}^{r} \sum_{j \in N_{lm_s} \backslash \{i\}} A_j^{lm_s}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\
& - \sum_{s=1}^{r} \sum_{k \in K} \hat{W}_k^{lm_s}\left(\bar{c}_{\left[N_{lm_s} \backslash C(N_{lm_s}, G_{N_l})\right]k} + \sum_{j \in C(N_{lm_s}, G_{N_l}) \backslash \{i\}} \bar{c}_{S(j, N_{lm_s})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\
& - \sum_{l' : N_{l'} \in sm(N_l; i)} H_{l'}\left(\bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right),
\end{aligned}
\tag{14}
$$

where for each $l \in \{1, \cdots, L\}$, each $m, m' \in \{1, \cdots, L_l\}$, and each $(c, E) \in \mathcal{D}$, $\hat{W}^{lm}\left(\cdot, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right)$ is additive,

$$
\begin{aligned}
& \sum_{i \in N_{lm}} A_i^{lm}\left(\bar{c}_{\cup s(N_l), \bar{c}, E}\right) + \sum_{k \in K} \hat{W}_k^{lm}\left(\bar{c}_{\cup s(N_l)k}, \bar{c}_{\cup s(N_l)k}, \bar{c}, E\right) \\
= & \sum_{i \in N_{lm'}} A_i^{lm'}\left(\bar{c}_{\cup s(N_l), \bar{c}, E}\right) + \sum_{k \in K} \hat{W}_k^{lm'}\left(\bar{c}_{\cup s(N_l)k}, \bar{c}_{\cup s(N_l)k}, \bar{c}, E\right),
\end{aligned}
\tag{15}
$$

and, $s(\cdot)$, $sm(\cdot)$, and $C^*(\cdot)$ are defined on the directed graph $\mathcal{G}(N_{l^*})$ with root $N_{l^*} \in \mathcal{N}$.

Now we are ready to state our main result.

Theorem 1. *Assume that G is a connected graph and $|N| \geq 3$. Then a rule over a rich domain satisfies reallocation-proofness if and only if it is a member of the HAW-family.*

The proof is in Appendix C.

By Lemma 2, we may replace *reallocation-proofness* in Theorem 1 with the combination of *pairwise reallocation-proofness* and *pairwise non-bossiness*.

Corollary 3. *Assume that G is a connected graph. Then a rule over a rich domain satisfies* pairwise reallocation-proofness *and* pairwise non-bossiness *if and only if it is a member of the HAW-family.*

It follows from Theorem 1 and Propositions 1-6 that:

Corollary 4. *Assume that G is a connected graph. Then the following two statements are equivalent*:
(i) *Graph G is rigid**;
(ii) *Reallocation-proofness under $\mathcal{C}(G)$ is equivalent to reallocation-proofness under the unrestricted coalition structure.*

A Structure of Connected Graph

In this section, we prove Lemmas 4 and 5. We begin with some useful facts on rigid graphs and rigid* graphs.

Fact 1. *When there are at least three nodes, rigidity* implies rigidity.*

Proof. Let $G \equiv (N, D)$ be rigid*. Assume $|N| \geq 3$. Suppose by contradiction that G is not rigid. Let $ij \in D$ be a connection edge. Then $G' \equiv (N, D \backslash \{ij\})$ is disconnected. Then since $|N| \geq 3$, i or j has an adjacent node in $N \backslash \{i, j\}$ on G'. Suppose that i has an adjacent node $h \in N \backslash \{i, j\}$ on G'. Then there is no path from h to j on G'. Since the set of edges of $G_{N \backslash \{i\}}$ is a subset of the set of edges of G', that is, $D \backslash \{ij\}$, then there is no path from h to j on $G_{N \backslash \{i\}}$ either. Thus $G_{N \backslash \{i\}}$ is disconnected. This shows that i is a connection node, contradicting rigidity* of G. ∎

Fact 2. *When $N \equiv \{i, j\}$ and $D \equiv \{ij\}$, $G \equiv (N, D)$ is rigid* but not rigid.*

Fact 3. *If G is rigid, $M \subseteq N$, and G_M is a maximal rigid* subgraph, then $|M| \geq 3$.*

Proof. Suppose $|M| = 1$, say $M = \{i\}$. Then because G is connected, there is $j \neq i$ such that $ij \in D$. Then $G_{\{i,j\}}$ is rigid*, contradicting the maximal rigidity of G_M. Suppose that $|M| = 2$, say, $M = \{i, j\}$. Let $G' \equiv (N, D \backslash \{ij\})$. Let M_i be the set of nodes connected with i on G' and M_j the set of nodes connected with j on G'. Since G is rigid, then ij is not a connection edge. So $M_i \cap M_j \neq \emptyset$. Let $h \in M_i \cap M_j$. Let $p(i,h)$ be a path in G'_{M_i} from i to h and $p(h,j)$ a path in G'_{M_j} from h to j. Let M' be the set of nodes in the two paths. Clearly, $M \subseteq M'$. Then $G_{M'}$ has a total cycle and so it is a rigid* graph, contradicting the maximal rigidity of G_M. ∎

Fact 4. *Let G be rigid. Let $M, M' \subseteq N$ be such that G_M and $G_{M'}$ are maximal rigid* subgraphs and $M \neq M'$. Then*
(i) *Either $|M \cap M'| = 0$ or 1.*
(ii) *If $i \in M \cap M'$, i is a connection node on G.*
(iii) *If $i \in M \cap M'$, $h \in M$, and $h' \in M'$, every path from h to h' contains i, that is, i is between h and h'.*

Proof. Proof of (i). Suppose by contradiction that $M \cap M'$ contains at least two nodes. For each $i \in M \backslash M'$, since i is not a connection node in G_M, $G_{M\backslash\{i\}}$ is connected. Since $i \notin M \cap M' \neq \emptyset$, every $j \in M\backslash\{i\}$ has a path to a node in $M \cap M'$, which has a path to any node in M'. Thus, $G_{(M\cup M')\backslash\{i\}}$ is connected. So i is not a connection node in $G_{M\cup M'}$. Similarly, we show that each $i \in M'\backslash M$ is not a connection node in $G_{M\cup M'}$. Now let $i \in M \cap M'$. Since $|M \cap M'| \geq 2$, there is $j \in (M \cap M')\backslash\{i\}$. Since both G_M and $G_{M'}$ are rigid*, both $G_{M\backslash\{i\}}$ and $G_{M'\backslash\{i\}}$ are connected. Because $j \in (M \cap M')\backslash\{i\}$, any node in $M\backslash\{i\}$ has a path, by way of j, to any node in $M'\backslash\{i\}$ on $G_{\{M\cup M'\}\backslash\{i\}}$. Hence $G_{\{M\cup M'\}\backslash\{i\}}$ is connected and i is not a connection node. This holds for each $i \in M \cap M'$. Evidently, each $i \in (M\backslash M') \cup (M'\backslash M)$ is not a connection node of $G_{M\cup M'}$ either. Therefore, $G_{M\cup M'}$ does not have any connection node and $G_{M\cup M'}$ is rigid*. This contradicts the maximal rigidity* of G_M.

Proof of (ii). Now let $i \in M \cap M'$. If i is not a connection node, $G_{N\backslash\{i\}}$ is connected. Pick $h \in M\backslash\{i\}$ and $h' \in M'\backslash\{i\}$. Then there is a path from h to h' on $G_{N\backslash\{i\}}$. Now combining this path with $M \cup M'$, we obtain a rigid* subgraph, contradicting the maximal rigidity* of G_M.

Proof of (iii). This follows easily from (ii). ∎

Clearly, if G has a total cycle, G is rigid*. There are, of course, rigid* graphs that have no total cycle. No tree with at least three nodes is rigid*.

Now we are ready to prove Lemma 4.

Proof of Lemma 4. Let $G \equiv (N, D)$ be a rigid graph.

Proof of part (i): We first show that N is divided into a finite number of subsets $N_1, \cdots, N_L$ with $\cup_{l=1}^{L} N_l = N$ such that for each $l = 1, \cdots, L$, $|N_l| \geq 3$

and G_{N_l} is a maximal rigid* subgraph on G. Pick a node $i \in N$. Find all maximal rigid* subgraphs containing i. Let $N_1, \cdots, N_m$ be the sets of nodes of these subgraphs. Then because of rigidity of G and Fact 3, $|N_1|, \cdots, |N_m| \geq 3$. If $\cup_{k=1}^{m} N_k = N$, we are done. Otherwise, since G is connected, pick $j \in N \setminus \cup_{k=1}^{m} N_k$ and find all maximal rigid* subgraphs containing j. Denote the sets of nodes of these subgraphs by $N_{m+1}, \cdots, N_{m+n}$. Then $|N_{m+1}|, \cdots, |N_{m+n}| \geq 3$. If $\cup_{k=1}^{m+n} N_k = N$, we are done. Otherwise, iterate the same procedure. Since N is finite, the iteration will end after a finite number of steps and, at the end, we get a list of subsets of N, $N_1, \cdots, N_L$, with the desired properties.

To prove the uniqueness, let $\{N_1, \cdots, N_L\}$ and $\{N'_1, \cdots, N_{L'}\}$ be two families of subsets of N satisfying the stated properties. Pick a node $i \in N$. Let $\{N_1, \cdots, N_m\}$ be the subfamily of elements in $\{N_1, \cdots, N_L\}$, which include i. Let $\{N'_1, \cdots, N'_{m'}\}$ be the subfamily of elements in $\{N'_1, \cdots, N_{L'}\}$, which include i. For each element N_k in the former subfamily, find $j \in N_k$ that is adjacent to i. Then there exists an element $N'_{k'}$ in the latter family which include both i and j (that is, ij is an edge of $G_{N'_{k'}}$). Therefore, by Fact 4, $N_k = N'_{k'}$. This shows $\{N_1, \cdots, N_m\} \subseteq \{N'_1, \cdots, N'_{m'}\}$. Similarly, we can show the reverse inclusion. Therefore, $\{N_1, \cdots, N_L\} = \{N'_1, \cdots, N_{L'}\}$.

Proof of part (ii): Suppose by contradiction that there exist $N_{l_1}, \cdots, N_{l_r} \in \{N_1, \cdots, N_L\}$ with $r \geq 3$ such that $N_{l_1} \cap N_{l_2} \neq \emptyset, \cdots, N_{l_{r-1}} \cap N_{l_r} \neq \emptyset$, and $N_{l_1} = N_{l_r}$. Then if we let $M \equiv N_{l_1} \cup \cdots \cup N_{l_r}$, G_M is rigid*. This contradicts the maximal rigidity* of G_{N_k} for each $k = 1, \cdots, r$. ∎

We use the next fact to prove Lemma 5.

Fact 5. *If G_M and $G_{M'}$ are maximal rigid subgraphs on G, then either $M = M'$ or $M \cap M' = \emptyset$.*

Proof. Let $M, M' \subseteq N$ be given as above. Assume $M \neq M'$. Suppose to the contrary $M \cap M' \neq \emptyset$. Since G_M has no connection edge disconnecting G_M and $M \cap M' \neq \emptyset$, there is no connection edge in G_M disconnecting $G_{M\cup M'}$. Similarly, there is no connection edge in $G_{M'}$ disconnecting $G_{M\cup M'}$. Therefore, $G_{M\cup M'}$ has no connection edge and so it is rigid. This contradicts maximal rigidity of G_M and $G_{M'}$. ∎

Fact 6. *Assume that $G \equiv (N, D)$ is a connected graph and that N is partitioned into a finite number of subsets $N_1, \cdots, N_L$ such that for each $l = 1, \cdots, L$, $|N_l| = 1$ or $|N_l| \geq 3$ and G_{N_l} is a maximal rigid subgraph on G. Then*
(i) *For each $l, l' = 1, \cdots, L$ with $l \neq l'$, there can be at most one edge $ii' \in D$ such*

that $i \in N_l$ and $i' \in N_{l'}$. If there is such an edge $ii' \in D$, it is a connection edge. (ii) *For each $l, l' = 1, \cdots, L$ with $l \neq l'$, if $i \in N_l$, $i' \in N_{l'}$, and $ii' \in D$, then for each $j \in N_l$ and each $j' \in N_{l'}$, every path from j to j' contains ii', that is, both i and i' are between j and j'.*

Proof. *Proof of part* (i): Let $l, l' \in \{1, \cdots, L\}$ be such that $l \neq l'$. Suppose to the contrary that at least two edges $ii', jj' \in D$ such that $i, j \in N_l$ and $i', j' \in N_{l'}$. Then any of these edges connecting N_l and $N_{l'}$ is not a connection edge on $G_{N_l \cup N_{l'}}$. Since neither G_{N_l} nor $G_{N_{l'}}$ has a connection edge, then no edge in G_{N_l} or $G_{N_{l'}}$ is a connection edge on $G_{N_l \cup N_{l'}}$. Therefore, $G_{N_l \cup N_{l'}}$ has no connection edge and so it is rigid. This contradicts to maximal rigidity of G_{N_l} and $G_{N_{l'}}$.

Now assume that $ii' \in D$ is such that $i \in N_l$ and $i' \in N_{l'}$. If ii' is not a connection edge, then we can find a path from a node in N_l to another node in $N_{l'}$, which does not include ii'. Now combining this path, N_l, and $N_{l'}$, we can construct a larger rigid subgraph than G_{N_l} and $G_{N_{l'}}$, contradicting maximal rigidity of G_{N_l} and $G_{N_{l'}}$.

Proof of part (ii): The proof follows directly from the definition of connection edge. ∎

Now we are ready to prove Lemma 5.

Proof of Lemma 5. Let $G \equiv (N, D)$ be a connected graph.

Proof of part (i): Since any edge is not a rigid subgraph, then if $M \subseteq N$ and G_M is rigid, either $|M| = 1$ or $|M| \geq 3$. The proof of the existence of a partition of N satisfying the property stated in part (i) is similar to the proof of part (i) in Lemma 4. The only difference is in showing that for any two subsets of N, $M \neq M'$, if G_M and $G_{M'}$ are maximal rigid subgraphs on G, then $M \cap M' = \emptyset$. This is shown in Fact 5.

To prove the uniqueness, let $\{N_1, \cdots, N_L\}$ and $\{N'_1, \cdots, N_{L'}\}$ be two partitions of N satisfying the stated properties. Pick a node $i \in N$. Without loss of generality, let N_l and $N'_{l'}$ be the members of the two partitions, which include i. Since $N_l \cap N'_{l'} \neq \emptyset$, then by Fact 5, $N_l = N'_{l'}$. Since this holds for every $i \in N$, the two partitions must be identical.

Proof of part (ii): Suppose by contradiction that there exist $r \geq 3$, $N_{l_1}, \cdots, N_{l_r} \in \{N_1, \cdots, N_L\}$, $i_1 \in N_{l_1}, \cdots, i_{r-1} \in N_{r-1}$, and $j_2 \in N_{l_2}, \cdots, j_r \in N_r$ such that $N_{l_1} = N_{l_r}$ and $i_1 j_2, i_2 j_3, \cdots, i_{r-1} j_r \in D$. Note that for each $s \in \{2, \cdots, r-2\}$, $i_s j_{s+1}$ connects N_{l_s} and $N_{l_{s+1}}$, and $i_{r-1} j_r$ connects N_{l_r} and N_{l_1}. Therefore, since each member of $\{N_{l_1}, \cdots, N_{l_r}\}$ is connected, then there is a path from i_1 to j_2

not containing $i_1j_2 \in D$. This means that deleting i_1j_2 does not disconnect G. So i_1j_2 is not a connection edge, contradicting part (i) of Fact 6. ∎

B Proof of Proposition 6

In this section, we prove Proposition 6.

Proof of Proposition 6. Let $G \equiv (N, D)$, $N_1, \cdots, N_L$, and $G_{N_1}, \cdots, G_{N_L}$ be given as in the proposition. Let f be a rule represented by a list of functions $(A^l \colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{N_l}, \hat{W}^l \colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K)_{l=1}^L$ as in (8)-(10). By Lemma 2, to show *reallocation-proofness* of f, we only have to show *pairwise reallocation-proofness* and *pairwise non-bossiness*. Let $ij \in D$ be an edge and $i, j \in N_l$ for some $l = 1, \cdots, L$. If $i, j \in N_l \backslash C(N_l)$, it follows directly from (8) and additivity of $\hat{W}^l(\cdot, \bar{c}, E)$ that the total award of i and j depends on c_i and c_j only through $c_i + c_j$ and $\bar{c}$. Thus, i and j cannot change their total award by a reallocation of c_i and c_j. Now consider the case that $i \in C(N_l)$ or $j \in C(N_l)$. Assume that $i \in C(N_l)$ and $j \in N \backslash C(N_l)$ (the same argument applies when j is also in $C(N_l)$). Let $\{G_{N_{l_1}}, \cdots, G_{N_{l_r}}\}$ is the set of all rigid* subgraphs other than G_{N_l}, to which i also belongs. Let $(c, E) \in \mathcal{D}$. By (8) and (9),

$$\begin{aligned}
f_i(c,E) + f_j(c,E) \;=\;& A_i^l(\bar{c},E) - \sum_{s=1}^{r} \sum_{h \in N_{l_s} \backslash \{i\}} A_h^{l_s}(\bar{c},E) \\
&+ \sum_{k \in K} \hat{W}_k^l\left(\bar{c}_{S(i,N_l)k}, \bar{c}, E\right) \\
&- \sum_{s=1}^{r} \sum_{k \in K} \hat{W}_k^{l_s}\left(\bar{c}_{[N_{l_s} \backslash C(N_{l_s})]k} + \sum_{h \in C(N_{l_s}) \backslash \{i\}} \bar{c}_{S(h,N_{l_s})k}, \bar{c}, E\right) \\
&+ A_j^l(\bar{c},E) + \sum_{k \in K} \hat{W}_k^l(c_{jk}, \bar{c}, E)
\end{aligned}$$

By additivity of $\hat{W}^l(\cdot, \bar{c}, E)$, this can be rewritten as follows:

$$\begin{aligned}
f_i(c,E) + f_j(c,E) \;=\;& A_i^l(\bar{c},E) + A_j^l(\bar{c},E) - \sum_{s=1}^{r} \sum_{h \in N_{l_s} \backslash \{i\}} A_h^{l_s}(\bar{c},E) \\
&+ \sum_{k \in K} \hat{W}_k^l\left(c_{ik} + c_{jk} + \bar{c}_{[S(i,N_l) \backslash \{i\}]k}, \bar{c}, E\right) \\
&- \sum_{s=1}^{r} \sum_{k \in K} \hat{W}_k^{l_s}\left(\bar{c}_{[N_{l_s} \backslash C(N_{l_s})]k} + \sum_{h \in C(N_{l_s}) \backslash \{i\}} \bar{c}_{S(h,N_{l_s})k}, \bar{c}, E\right).
\end{aligned}$$

Therefore, the total award of i and j depends on c_i and c_j only through $c_i + c_j$ and $\bar{c}$. This shows that f satisfies *pairwise reallocation-proofness*. *Pairwise non-bossiness* follows from the fact that in (8) and (9), the characteristic vectors of each link $ij \in D$, c_i and c_j, affect the awards of others only through $c_i + c_j$ and $\bar{c}$.

To prove the converse, let f be a rule satisfying *reallocation-proofness*. Then by Lemma 2, f satisfies *non-bossiness*. Consider N_1 and rigid* subgraph G_{N_1}. Let $\mathcal{D}_{N_1} \equiv \{(d, E) \in \mathbb{R}_+^{N_1 \times K} \times \mathbb{R}_{++} : \text{for some } (c, E) \in \mathcal{D},\ c_{N_1 \backslash C(N_1)} = d_{N_1 \backslash C(N_1)}$ and for each $i \in C(N_1)$, $\bar{c}_{S(i,N_1)} = d_i\}$. Let $g \colon \mathcal{D}_{N_1} \to \mathbb{R}^{N_1}$ be defined as follows: for each $(d, E) \in \mathcal{D}_{N_1}$,

$$g_i(d, E) \equiv \begin{cases} f_i(c, E) \text{ if } i \in N_1 \backslash C(N_1), \\ \sum\limits_{j \in S(i,N_1)} f_j(c, E) \text{ if } i \in C(N_1), \end{cases}$$

where $(c, E) \in \mathcal{D}$ is such that $c_{N_1 \backslash C(N_1)} = d_{N_1 \backslash C(N_1)}$ and for each $i \in C(N_1)$, $\bar{c}_{S(i,N_1)} = d_i$. To show that g is well-defined, let c, c' be such that $c_{N_1 \backslash C(N_1)} = c'_{N_1 \backslash C(N_1)} = d_{N_1 \backslash C(N_1)}$ and for each $i \in C(N_1)$, $\bar{c}_{S(i,N_1)} = \bar{c}'_{S(i,N_1)} = d_i$. For each $i \in C(N_1)$, if coalition $S(i, N_1)$ changes $c_{S(i,N_1)}$ to $c'_{S(i,N_1)}$, then since $S(i, N_1)$ is connected, by *reallocation-proofness* and *non-bossiness*, the total award of $S(i, N_1)$ remains constant and the awards of all others also remain constant. After making these changes for all agents in $C(N_1)$, we finally get c'. And for each $i \in C(N_1)$, the total award of $S(i, N_1)$ remains constant and the awards of all agents in $N_1 \backslash C(N_1)$ also remain constant. Therefore,

$$f_i(c, E) = f_i(c', E), \text{ if } i \in N_1 \backslash C(N_1);$$
$$\sum_{j \in S(i,N_1)} f_j(c, E) = \sum_{j \in S(i,N_1)} f_j(c', E), \text{ if } i \in C(N_1).$$

This shows that g is well-defined.

We now show that g is a rule over $\mathcal{D}_{N_1}$ satisfying *pairwise reallocation-proofness* and *pairwise non-bossiness* under $\mathcal{C}(G_{N_1})$ and, therefore, satisfying *reallocation-proofness* under $\mathcal{C}(G_{N_1})$. Let $i^*, j^* \in N_1 \backslash C(N_1)$ be such that $i^* j^* \in D_{N_1}$. Then it follows from *pairwise reallocation-proofness* and *pairwise non-bossiness* of f and the definition of g that this pair $\{i^*, j^*\}$ cannot change their total award or awards of others by any reallocation of characteristic vectors among the pair. Now consider a pair $\{i^*, j^*\}$ that is an edge in D_{N_1} and $i^* \in C(N_1)$. Let $(d, E), (d', E) \in \mathcal{D}_{N_1}$ be such that $d_{N_1 \backslash \{i^*, j^*\}} = d'_{N_1 \backslash \{i^*, j^*\}}$ and $d_{i^*} + d_{j^*} = d'_{i^*} + d'_{j^*}$. Let $c \in \mathcal{D}$ be such that $c_{N_1 \backslash C(N_1)} = d_{N_1 \backslash C(N_1)}$ and for each $i \in C(N_1)$, $\bar{c}_{S(i,N_1)} = d_i$. Without loss of generality, suppose $j^* \notin C(N_1)$ (a similar argument applies when $j^* \in C(N_1)$). Let c' be such that $\bar{c}'_{S(i^*,N_1)} = d'_{i^*}$ and $c'_{j^*} = d'_{j^*}$ and for each $i \notin S(i^*, N_1) \cup \{j^*\}$, $c'_i = c_i$. Then $\bar{c}'_{S(i^*,N_1)} + c'_{j^*} = \bar{c}_{S(i^*,N_1)} + c_{j^*}$ and $c'_{N\setminus(S(i^*,N_1)\cup\{j^*\})} = c_{N\setminus(S(i^*,N_1)\cup\{j^*\})}$. Since i^*j^* is an edge, $S(i^*, N_1) \cup \{j^*\}$ is connected. Thus by *reallocation-proofness* and *non-bossiness* of f,

$$\begin{aligned} \sum_{i\in S(i^*,N_1)\cup\{j^*\}} f_i(c', E) &= \sum_{i\in S(i^*,N_1)\cup\{j^*\}} f_i(c, E); \\ f_{N\setminus(S(i^*,N_1)\cup\{j^*\})}(c', E) &= f_{N\setminus(S(i^*,N_1)\cup\{j^*\})}(c, E). \end{aligned}$$

Therefore,

$$\begin{aligned} g_{i^*}(d', E) + g_{j^*}(d', E) &= g_{i^*}(d, E) + g_{j^*}(d, E); \\ g_{N\setminus\{i^*,j^*\}}(c', E) &= g_{N\setminus\{i^*,j^*\}}(c, E). \end{aligned}$$

This shows that g satisfies *pairwise reallocation-proofness* and *pairwise non-bossiness* under $\mathcal{C}(G_{N_1})$.

Since G_{N_1} is rigid* and $|N_1| \geq 3$, then applying Proposition 5, we conclude that there exist $A^1 \colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{N_1}$ and $\hat{W}^1 \colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that for each $(d, E) \in \mathcal{D}_{N_1}$ and each $i \in N_1$,

$$g_i(d, E) = A^1_i(\bar{d}, E) + \sum_{k\in K} \hat{W}^1_k(d_{ik}, \bar{d}, E),$$

and $\hat{W}^1\left(\cdot, \bar{d}, E\right)$ is additive. Therefore, for each $(c, E) \in \mathcal{D}$,

$$\begin{aligned} f_i(c, E) &= A^1_i(\bar{c}, E) + \sum_{k\in K} \hat{W}^1_k(c_{ik}, \bar{c}, E) \text{ if } i \in N_1\setminus C(N_1); \\ \sum_{j\in S(i,N_1)} f_j(c, E) &= A^1_i(\bar{c}, E) + \sum_{k\in K} \hat{W}^1_k\left(\bar{c}_{S(i,N_1)k}, \bar{c}, E\right) \text{ if } i \in C(N_1). \end{aligned}$$

Applying this argument for each rigid* subgraph G_{N_l}, $l = 1, \cdots, L$, we obtain a list of functions $\left(A^l \colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{N_l}, \hat{W}^l \colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K\right)_{l=1}^L$ such that for each $(c, E) \in \mathcal{D}$ and each $l = 1, \cdots, L$,

$$\begin{aligned} f_i(c, E) &= A^l_i(\bar{c}, E) + \sum_{k\in K} \hat{W}^l_k(c_{ik}, \bar{c}, E) \text{ if } i \in N_l\setminus C(N_l); && (\dagger) \\ \sum_{j\in S(i,N_l)} f_j(c, E) &= A^l_i(\bar{c}, E) + \sum_{k\in K} \hat{W}^l_k\left(\bar{c}_{S(i,N_l)k}, \bar{c}, E\right) \text{ if } i \in C(N_l), && (\ddagger) \end{aligned}$$

and $\hat{W}^l(\cdot, \bar{c}, E)$ is additive.

Assume that $\{G_{N_{l_1}}, \cdots, G_{N_{l_r}}\}$ is the set of all rigid* subgraphs other than G_{N_l}, to which i also belongs. Using (†) and (‡), we obtain

$$\begin{aligned} f_i(c,E) &= A_i^l(\bar{c},E) + \sum_{k\in K} \hat{W}_k^l\left(\bar{c}_{S(i,N_l)k},\bar{c},E\right) \\ &\quad - \sum_{s=1}^{r} \sum_{j\in N_{l_s}\setminus C(N_{l_s})} \left(A_j^{l_s}(\bar{c},E) + \sum_{k\in K} \hat{W}_k^{l_s}(c_{jk},\bar{c},E)\right) \\ &\quad - \sum_{s=1}^{r} \sum_{j\in C(N_{l_s})\setminus\{i\}} \sum_{j'\in S(j,N_{l_s})} f_{j'}(c,E). \end{aligned}$$

Again by (‡) and additivity of $\hat{W}^{l_s}(\cdot,\bar{c},E)$,

$$\begin{aligned} f_i(c,E) &= A_i^l(\bar{c},E) + \sum_{k\in K} \hat{W}_k^l\left(\bar{c}_{S(i,N_l)k},\bar{c},E\right) \\ &\quad - \sum_{s=1}^{r} \sum_{j\in N_{l_s}\setminus C(N_{l_s})} A_j^{l_s}(\bar{c},E) - \sum_{s=1}^{r}\sum_{k\in K} \hat{W}_k^{l_s}\left(\bar{c}_{[N_{l_s}\setminus C(N_{l_s})]k},\bar{c},E\right) \\ &\quad - \sum_{s=1}^{r} \sum_{j\in C(N_{l_s})\setminus\{i\}} A_j^{l_s}(\bar{c},E) - \sum_{s=1}^{r}\sum_{k\in K} \hat{W}_k^{l_s}\left(\sum_{j\in C(N_{l_s})\setminus\{i\}} \bar{c}_{S(j,N_{l_s})k},\bar{c},E\right). \end{aligned}$$

Since $\hat{W}^{l_s}(\cdot,\bar{c},E)$ is additive for each $l_s \in \{l_1,\cdots,l_r\}$,

$$\begin{aligned} f_i(c,E) &= A_i^l(\bar{c},E) - \sum_{s=1}^{r} \sum_{j\in N_{l_s}\setminus\{i\}} A_j^{l_s}(\bar{c},E) \\ &\quad + \sum_{k\in K} \hat{W}_k^l\left(\bar{c}_{S(i,N_l)k},\bar{c},E\right) \\ &\quad - \sum_{s=1}^{r}\sum_{k\in K} \hat{W}_k^{l_s}\left(\bar{c}_{[N_{l_s}\setminus C(N_{l_s})]k} + \sum_{j\in C(N_{l_s})\setminus\{i\}} \bar{c}_{S(j,N_{l_s})k},\bar{c},E\right) \end{aligned}$$

Applying the same argument for l_1,

$$\begin{aligned} f_i(c,E) &= A_i^{l_1}(\bar{c},E) - \sum_{j\in N_l\setminus\{i\}} A_j^l(\bar{c},E) - \sum_{s=2}^{r} \sum_{j\in N_{l_s}\setminus\{i\}} A_j^{l_s}(\bar{c},E) \\ &\quad + \sum_{k\in K} \hat{W}_k^{l_1}\left(\bar{c}_{S(i,N_{l_1})k},\bar{c},E\right) \\ &\quad - \sum_{k\in K} \hat{W}_k^{l}\left(\bar{c}_{[N_l\setminus C(N_l)]k} + \sum_{j\in C(N_l)\setminus\{i\}} \bar{c}_{S(j,N_l)k},\bar{c},E\right) \\ &\quad - \sum_{s=2}^{r}\sum_{k\in K} \hat{W}_k^{l_s}\left(\bar{c}_{[N_{l_s}\setminus C(N_{l_s})]k} + \sum_{j\in C(N_{l_s})\setminus\{i\}} \bar{c}_{S(j,N_{l_s})k},\bar{c},E\right). \end{aligned}$$

In order for $f(\cdot)$ to be well-defined, the right-hand sides of the two expressions for $f_i(c,E)$ should be equal. This equality is equivalent to

$$\begin{aligned} & A_i^l(\bar{c},E) + \sum_{j \in N_l \setminus \{i\}} A_j^l(\bar{c},E) + \sum_{k \in K} \hat{W}_k^l \left(\bar{c}_{[N_l \setminus C(N_l)]k} + \sum_{j \in C(N_l)} \bar{c}_{S(j,G_{N_l})k}, \bar{c}, E \right) \\ = \; & A_i^{l_1}(\bar{c},E) + \sum_{j \in N_{l_1} \setminus \{i\}} A_j^{l_1}(\bar{c},E) + \sum_{k \in K} \hat{W}_k^{l_1} \left(\bar{c}_{\left[N_{l_1} \setminus C(N_{l_1})\right]k} + \sum_{j \in C(N_{l_1})} \bar{c}_{S(j,N_{l_1})k}, \bar{c}, E \right). \end{aligned}$$

Since for each $k \in K$,

$$\bar{c}_{[N_l \setminus C(N_l)]k} + \sum_{j \in C(N_l)} \bar{c}_{S(j,N_l)k} = \bar{c}_{\left[N_{l_1} \setminus C(N_{l_1})\right]k} + \sum_{j \in C(N_{l_1})} \bar{c}_{S(j,N_{l_1})k} = \bar{c}_k,$$

we obtain (10). ∎

C Proof of Theorem 1

In this section, we prove Theorem 1.

Proof of Theorem 1. We skip the proof that every rule in HAW-family satisfies *reallocation-proofness*, since it can be done using the same arguments as in the proofs of Propositions 3 and 6.

Let $G \equiv (N,D)$ be a connected graph. Let $\mathcal{N} \equiv \{N_1, \cdots, N_L\}$ and $\mathcal{R} \equiv \{G_{N_1}, \cdots, G_{N_L}\}$. By Lemma 5, for each $l = 1, \cdots, L$, $|N_l| = 1$ or $|N_l| \geq 3$. By Lemma 4, for each $l = 1, \cdots, L$, N_l is again divided into L_l subsets, $L_l \in \mathbb{N}$, $N_{l1}, \cdots, N_{lL_l}$ such that $\cup_{m=1}^{L_l} N_{lm} = N_l$ and for each $m = 1, \cdots, L_l$, $G_{N_{lm}}$ is a maximal rigid* subgraph on G_{N_l}. Let $\mathcal{G} \equiv (\mathcal{N}, \mathcal{E})$ be the graph in Definition 4. Pick $l^* \in \{1, \cdots, L\}$ and consider the directed tree $\mathcal{G}(N_{l^*})$. Roughly speaking the following proof is the combination of the arguments used in the proofs of Propositions 3 and 6.

Let f be a rule satisfying *reallocation-proofness*. Then by Lemma 2, f satisfies *non-bossiness*. Define a function $H\colon \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^L$ such that for each $l \in L$ and each $(x,y,E) \in \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++}$,

$$H_l(x,y,E) \equiv \sum_{i \in \cup s(N_l)} f_i(c,E),$$

for some $(c,E) \in \mathcal{D}$ with $\sum_{i \in \cup s(N_l)} c_i = x$ and $\bar{c} = y$. For all other (x,y,E), define $H_l(x,y,E)$ arbitrarily. Since both $\cup s(N_l)$ and $N \setminus \cup s(N_l)$ are connected in G, then by *reallocation-proofness* and *non-bossiness*, we can show that $H(\cdot)$ is well-defined.

Let $l \in \{1, \cdots, L\}$. If $|N_l|$ is a singleton and so $L_l = 1$, then let A^{l1} and $\hat{W}^{l1}$ be such that for each $(c, E) \in \mathcal{D}$, $A_i^{l1}(\bar{c}, E) + \sum_{k \in K} \hat{W}_k^{l1}(c_{ik}, \bar{c}, E) = H_l\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) - \sum_{l': N_{l'} \in sm(N_l)} H_{l'}(\bar{c}_{\cup s(N_{l'})}, \bar{c}, E)$, where $i \in N_l$. Then (13) holds and the remaining three equations (11), (12), and (14) hold vacuously.

Now consider the case when $|N_l| \geq 3$. Fix $y \in \mathbb{R}^K_{++}$. Let $\mathcal{D}_{N_l}(y) \equiv \{(d, E) \in \mathbb{R}^{N_l \times K}_{++} \times \mathbb{R}_{++}$: for some $(c, E) \in \mathcal{D}$, $\bar{c} = y$, $c_{N_l \setminus C^*(N_l)} = d_{N_l \setminus C^*(N_l)}$, and for each $i \in C^*(N_l)$, $c_i + \bar{c}_{\cup s^0(N_l;i)} = d_i\}$. Define $g \colon \mathcal{D}_{N_l}(y) \to \mathbb{R}^{N_l}$ as follows: for each $(d, E) \in \mathcal{D}_{N_l}(y)$ and each $i \in N_l$,

$$g_i(d, E) \equiv \begin{cases} f_i(c, E), & \text{if } i \notin C^*(N_l), \\ f_i(c, E) + \sum_{j \in \cup s^o(N_l;i)} f_j(c, E), & \text{if } i \in C^*(N_l), \end{cases}$$

for some $(c, E) \in \mathcal{D}$ such that $\bar{c} = y$, $\bar{c}_{\cup s(N_l)} = d$, $c_{N_l \setminus C^*(N_l)} = d_{N_l \setminus C^*(N_l)}$, and for each $i \in C^*(N_l)$, $c_i + \bar{c}_{\cup s^0(N_l;i)} = d_i$. We now show that g is well-defined. Let $(c, E) \in \mathcal{D}$ and $c' \in \mathbb{R}^{N \times K}_+$ be such that $\bar{c} = \bar{c}' = y$, $\bar{c}_{\cup s(N_l)} = \bar{c}'_{\cup s(N_l)} = d$, $c_{N_l \setminus C^*(N_l)} = c'_{N_l \setminus C^*(N_l)} = d_{N_l \setminus C^*(N_l)}$, and for each $i \in C^*(N_l)$, $c_i + \bar{c}_{\cup s^0(N_l;i)} = c'_i + \bar{c}'_{\cup s^0(N_l;i)} = d_i$. Since $N \setminus \cup s(N_l)$ is connected, then by *reallocation-proofness* and *non-bossiness*, $c_{N \setminus \cup s(N_l)}$ is irrelevant in this definition. So without loss of generality, we may assume that $c_{N \setminus \cup s(N_l)} = c'_{N \setminus \cup s(N_l)}$. For each $i \in C^*(N_l)$, let $T_i \equiv \{i\} \cup [\cup s^0(N_l; i)]$. Then T_i is connected. So by *reallocation-proofness* and *non-bossiness*, if coalition T_i changes c_{T_i} to c'_{T_i}, then the total award of T_i and the awards of all others in $N \setminus T_i$ do not change. After making these changes for all $i \in C^*(N_l)$, we end up with c' and, throughout this process, the total award of coalition $T_i = \{i\} \cup [\cup s^0(N_l; i)]$ for each $i \in C^*(N_l)$, and the awards for all $j \in N_l \setminus C^*(N_l)$ do not change. Therefore, for each $i \in N_l \setminus C^*(N_l)$, $f_i(c, E) = f_i(c', E)$, and for each $i \in C^*(N_l)$, $f_i(c, E) + \sum_{j \in \cup s^o(N_l;i)} f_j(c, E) = f_i(c', E) + \sum_{j \in \cup s^o(N_l;i)} f_j(c', E)$.

We now show that g is a rule over $\mathcal{D}_{N_l}(y)$ satisfying *pairwise reallocation-proofness* and *pairwise non-bossiness* under $\mathcal{C}(G_{N_l})$ and, therefore, *reallocation-proofness* under $\mathcal{C}(G_{N_l})$. Let $i^*, j^* \in N_l \setminus C^*(N_l)$ be such that $i^* j^* \in D_{N_l}$. Then it follows from *pairwise reallocation-proofness* and *pairwise non-bossiness* of f and the definition of g that this pair $\{i^*, j^*\}$ cannot change their total award or awards of others by any reallocation of characteristic vectors among the pair. Now consider a pair $\{i^*, j^*\}$ that is an edge in D_{N_l} and $i^* \in C^*(N_l)$. Let $(d, E), (d', E) \in \mathcal{D}_{N_l}(y)$ be such that $d_{N_l \setminus \{i^*, j^*\}} = d'_{N_l \setminus \{i^*, j^*\}}$ and $d_{i^*} + d_{j^*} = d'_{i^*} + d'_{j^*}$. Let $c \in \mathbb{R}^{N \times K}_+$ be such that $\bar{c} = y$, $\bar{c}_{\cup s(N_l)} = \bar{d}$, $c_{N_l \setminus C^*(N_l)} = d_{N_l \setminus C^*(N_l)}$, and for each $i \in C^*(N_l)$, $c_i + \bar{c}_{\cup s^0(N_l;i)} = d_i$. Without loss of generality, suppose

$j^* \notin C^*(N_l)$ (a similar argument applies when $j^* \in C^*(N_l)$). Let $c' \in \mathbb{R}_+^{N \times K}$ be such that $c'_{N\backslash[\{i^*,j^*\}\cup[\cup s^0(N_l;i^*)]]} = c_{N\backslash[\{i^*,j^*\}\cup[\cup s^0(N_l;i^*)]]}$, $c'_{i^*} + \bar{c}'_{\cup s^0(N_l;i^*)} = d'_{i^*}$, and $c'_{j^*} = d'_{j^*}$. Since $d_{i^*} + d_{j^*} = d'_{i^*} + d'_{j^*}$, $c_{i^*} + \bar{c}_{\cup s^0(N_l;i^*)} + c_{j^*} = c'_{i^*} + \bar{c}'_{\cup s^0(N_l;i^*)} + c'_{j^*}$. Since i^*j^* is an edge and $\{i^*\} \cup [\cup s^0(N_l;i^*)]$ is connected, $\{i^*, j^*\} \cup [\cup s^0(N_l;i^*)]$ is connected. Thus by *reallocation-proofness* and *non-bossiness* of f,

$$\begin{aligned}
\sum_{i\in\{i^*\}\cup[\cup s^0(N_l;i^*)]} f_i(c',E) + f_{j^*}(c',E) &= \sum_{i\in\{i^*\}\cup[\cup s^0(N_l;i^*)]} f_i(c,E) + f_{j^*}(c,E);\\
f_{N\backslash(\{i^*,j^*\}\cup[\cup s^0(N_l;i^*)])}(c',E) &= f_{N\backslash(\{i^*,j^*\}\cup[\cup s^0(N_l;i^*)])}(c,E).
\end{aligned}$$

Therefore,

$$\begin{aligned}
g_{i^*}(d',E) + g_{j^*}(d',E) &= g_{i^*}(d,E) + g_{j^*}(d,E);\\
g_{N\backslash\{i^*,j^*\}}(c',E) &= g_{N\backslash\{i^*,j^*\}}(c,E).
\end{aligned}$$

This shows that g satisfies *pairwise reallocation-proofness* and *pairwise non-bossiness* under $\mathcal{C}(G_{N_l})$.

Now applying Proposition 6 and the definition of $H(\cdot)$, we conclude that there exists a list of functions $\left(A^m\colon \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^{N_{lm}}, \hat{W}^m\colon \mathbb{R}_+ \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^K\right)_{m=1}^{L_l}$ such that for each $(d,E) \in \mathcal{D}_{N_l}(y)$, each $m \in \{1,\cdots,L_l\}$, and each $i \in N_{lm}$, if $i \in N_{lm}\backslash C(N_{lm}, G_{N_l})$,

$$g_i(c,E) = A_i^m(\bar{d},E) + \sum_{k\in K} \hat{W}_k^m(c_{ik},\bar{d},E);$$

and if $i \in C(N_{lm}, G_{N_l})$ and $\{G_{N_{lm_1}},\cdots,G_{N_{lm_r}}\}$ is the set of all rigid* subgraphs other than $G_{N_{lm}}$, to which i also belongs,

$$\begin{aligned}
g_i(c,E) &= A_i^m\left(\bar{d},E\right) - \sum_{s=1}^{r}\sum_{j\in N_{lm_s}\backslash\{i\}} A_j^{m_s}(\bar{d},E) + \sum_{k\in K}\hat{W}_k^m\left(\bar{d}_{S(i,N_{lm})k},\bar{d},E\right)\\
&\quad - \sum_{s=1}^{r}\sum_{k\in K}\hat{W}_k^{m_s}\left(\bar{d}_{\left[N_{lm_x}\backslash C(N_{lm_s},G_{N_l})\right]k} + \sum_{j\in C(N_{lm_s},G_{N_l})\backslash\{i\}} \bar{d}_{S(j,N_{lm_s})k},\bar{d},E\right),
\end{aligned}$$

where for each $l \in L$, $\hat{W}^m\left(\cdot,\bar{d},E\right)$ is additive and satisfies (10). Now for each $m \in \{1,\cdots,L_l\}$, let $A^{lm}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right) \equiv A^m\left(\bar{c}_{\cup s(N_l)},E\right)$ and for each $k \in K$, $\hat{W}_k^{lm}\left(\cdot,\bar{c}_{\cup s(N_l)},\bar{c},E\right) \equiv \hat{W}_k^m\left(\cdot,\bar{c}_{\cup s(N_l)},E\right)$. Then by definition of $H(\cdot)$, we obtain (11)-(14). We obtain (15) from (10). ∎

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