Student Placement to Public Schools in US: Two New Solutions

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Abstract

An increasingly popular practice in placing students to public schools in the US is the use of school choice programs. In a school choice program, each student submits a *preference list* of schools to a central authority which then decides upon how to place students to schools while also taking the *priorities* of students for schools into consideration. The crucial issue here is determining a central student placement mechanism that would ensure (1) equity (*i.e.*, students' priorities should be respected), (2) optimality, and (3) immunity to preference manipulation. We propse two new competing mechanisms as promising alternatives to those mechanisms that are both currently in use and that have been proposed in the existing literature. The first mechanism we propose eliminates the potentially large efficiency loss in the well-known Gale-Shapley *student optimal stable machanism*, while achieving equity and immunity to preference manipulation to quite a satisfactory degree. The second mechanism we propose considerably improves the equity aspects of the *top trading cycles mechanism* of Abdulkadiroğlu and Sönmez (*Amer. Econ. Rev.* 93, 2003, pp. 729-747) without paying any cost.

Keywords: Student placement problem; Efficiency adjusted deferred acceptance mechanism; Equitable top trading cycles mechanism.

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1 Introduction

Until recently, students in the US were placed to public schools based on their residence area, *i.e.,* a student had to go to the school whichever was closest to his/her home. Such a practice caused inequalities among students of the rich and poor families since the rich always had the opportunity to choose the place they wanted to live. Starting in 1987 with Minnesota, many US states began adopting school choice programs which give each student the option to choose the school he/she wants to attend. In a school choice program, each student submits a list of *preferences* of schools to a central placement authority, such as the school district, which then decides as to which student will be placed to which school. In addition to the preferences of students, one other major factor that the central placement authority takes into consideration is the *priorities* of students for schools. There may be several criteria in determining a priority order for a school. For example, in Boston, the first priority for a school is given to the students who are in the same walk zone and who have a sibling attending that school, the second priority to those who only have a sibling attending that school, the third priority to those who are only in the same walk zone, and the fourth priority to other students. (For those students who are in the same priority group, the order is determined by a lottery.) Today, many cities such as Boston, Seattle, Minneapolis, (parts of) New York, and Columbus are using school choice programs and it is strongly anticipated that in the near future there will not be any school district which does not utilize a school choice program.

Given the significance of school choice programs, the importance of the way one will employ for the placement of students to schools is clearly obvious. In this paper, we consider this issue from a mechanism design perspective. A *student placement problem* is a pair consisting of a *preference profile* of students and a collection of *priority orders* for schools. For a given student placement problem, at an *allocation* each student is placed to only one school and the number of students placed to a particular school does not exceed the number of available seats at that school. An allocation is *Pareto efficient* if there is no other allocation which makes all students at least as well off and at least one student better off.

A very closely related problem to the student placement problem is the well-known *college admissions problem* due to Gale and Shapley (1962). The main difference between the two problems is that in a college admissions problem, the priorities of schools are replaced by the preferences of schools.

Although priorities and preferences are mathematically equivalent, unlike priorities, preferences call for welfare and strategic considerations. Here, priorities of schools are enforced by local/state laws and no school has a say on the way its priority order is determined. In a college admissions problem, an allocation is *stable* if there is no student-school pair (i, s) such that student i prefers school s to the school he is placed to, and school s prefers student i to at least one student who is placed to it. The natural counterpart of stability in our context is "fairness." We say that student i has *justified envy* for a student at school s at an allocation if he prefers school s to the school he is placed to, and he has higher priority for school s than at least one student who is placed to it. An allocation is *fair* if there is no student-school pair (i, s) such that student i has justified envy for a student who is placed to school s.

A *student placement mechanism* or simply, a mechanism, is a systematic way of selecting an allocation for each student placement problem. A student placement mechanism is Pareto efficient if it always selects Pareto efficient allocations. A student placement mechanism is fair if it always selects fair allocations. It is by now well-known that for a given student placement problem, there may not exist a Pareto efficient and fair allocation. Consequently, there is no mechanism that is both fair and Pareto efficient.

In a very recent paper, Abdulkadiroğlu and Sönmez (2003), for the first time in the literature, approach the student placement problem from a mechanism design perspective. They examine some of the real-life student placement mechanisms and point out some serious deficiencies they have. (See Section 2 for one such frequently-used real-life mechanism.) They come up with two new mechanisms to fix these deficiencies.

The first one is the Gale and Shapley's (1962) *student optimal stable mechanism* (SOSM) proposed for the college admissions problem. Following Abdulkadiroğlu and Sönmez (2003), in consultation with Alvin Roth, the New York education department has started using this mechanism for the placement of ninth graders to public schools in the Fall of 2003. SOSM is fair and it Pareto dominates any other fair mechanism. It is also strategy-proof. However, it is not Pareto efficient.

The second mechanism Abdulkadiroğlu and Sönmez (2003) advocate is based on an adaptation of *Gale's top trading cycles procedure* to the student placement context. They call this mechanism the *top trading cycles mechanism* (TTCM). TTCM is Pareto efficient and strategy-proof. However, it is not fair.

There are three most important properties one can expect a good student placement mechanism to satisfy: (1) Equity (*e.g.*, fairness), (2) Optimality (*e.g.*, Pareto efficiency), and (3) Immunity to strategic behavior (*e.g.*, strategy-proofness). In this context, it is impossible to fully achieve all three. Let that alone, even fairness and Pareto efficiency are incompatible. If one totally gives up on one of the three properties, then it is possible to fully achieve the other two. However, such a sacrifice may lead to unacceptable situations in terms of the property that is given up. For example, if one insists on the use of a fair and strategy-proof mechanism, then SOSM is the obvious choice as it Pareto dominates any other fair mechanism (and it is strategy-proof). However, the outcome of this mechanism could be so inefficient that most of the students may end up at their almost worst choices (Example 2). On the other hand, if one values Pareto efficiency and strategy-proofness above any other pair, then a choice like TTCM may result in situations where equity is severely violated. For example, for a given student placement problem, TTCM may not select the Pareto efficient and fair allocation, even if such an allocation exists (Example 3), and thereby introduce unwarranted justified envy among students. Or, the TTCM outcome may not Pareto dominate SOSM outcome, even if SOSM outcome is Pareto inefficient (Example 4). Our first mechanism is motivated by these observations.

First, we propose a new equity criterion as an alternative to the strong requirement "fairness." Consider an allocation at which there is a student who has justified envy for another student who is placed to a certain school. That is, there is a student who would rather be placed to another school for which he has higher priority than another student who is currently placed to that school. Naturally, such a student would consider this allocation as unfair. Consequently, he would object to that allocation and demand a fair allocation to be decided upon. Note that when this student objects to the earlier allocation, he has the hope that at the new allocation, he will be placed to that school or, to an even better school for him. If there is indeed an allocation that does this, then the point this student makes is reasonable. But, what if there is no such allocation? That is, what if there is no fair allocation that places this student to either that school or to a better school for him. In such a case, the desire of this student can not be fulfilled, and thus it is groundless. The alternative equity criterion we propose is based on this observation. We say that an allocation is *reasonably fair* for a student placement problem if whenever a student has justified envy for another student who is placed to a certain school, then there is no fair allocation that places him to that school or to any school better for him than that school. A reasonably fair mechanism always selects reasonably fair allocations.

We propose a modification of SOSM that takes care of the serious efficiency loss in this mechanism. In many ways, it mimics SOSM. We call this mechanism the *efficiency adjusted deferred acceptance mechanism* (EADAM). EADAM is reasonably fair (Proposition 7). Furthermore, it is Pareto efficient (Proposition 8). Yet, it is not immune to strategic manipulation. However, there is no mechanism that is reasonably fair, Pareto efficient, and strategy-proof (Proposition 9). Because a mechanism is not strategy-proof does not necessarily mean that it can easily be manipulated. Based on a result due to Ehlers (2002b), we show that EADAM is *practically* strategy-proof (Proposition 10). EADAM achieves all three important considerations to quite a satisfactory degree, which in turn gives it the edge over its competitors in terms of practical applicability.

The second mechanism we propose is another adaptation of Gale's top trading cycles procedure. In our approach, unlike the TTCM of Abdulkadiroğlu and Sönmez (2003), instead of giving all the trading power to those students with the highest priority for a school, we distribute the trading rights of seats for each school among those who are entitled one seat at that school and allow them to trade in such a way so as situations of justified envy are avoided as much as possible. We call this mechanism the *equitable top trading cycles mechanism* (ETTCM). ETTCM is also Pareto efficient and strategy-proof like TTCM. Furthermore, it considerably improves the equity aspects of TTCM without paying any cost.

We believe that both mechanisms we propose will contribute significantly to the mechanism design literature and that they will be powerful tools for overcoming the shortcomings of both the currently in use real-life mechanisms and the so far proposed alternative mechanisms in the literature.

The paper is organized as follows. In Section 2 we model the student placement problem and describe a popular real-life mechanism (the Boston mechanism), the student optimal stable mechanism, and the top trading cycles mechanism. In Section 3 we motivate and propose the new equity criterion "reasonable fairness." In Sections 4 and 5 we introduce the efficiency adjusted deferred acceptance mechanism and the equitable top trading cycles mechanism respectively. The paper contains an Appendix at the end.

2 Student Placement Problem

In a student placement problem, there is a certain number of students each of whom are to be placed to a school among a certain number of schools. Each school has a certain number of available seats and the total number of seats is greater than the number of students. Let us denote the set of students by $I \equiv \{i_1, i_2, \ldots, i_n\}$. A generic element in I is denoted by i. Let us denote the set of schools by $S \equiv \{s_1, s_2, \ldots, s_m\}$. A generic element in S is denoted by s. Each school has a certain number of available seats. For each school there is a strict priority order of all students, and each student has strict preferences over all schools. The priority orders are determined according to state/local laws and certain criteria of school districts. Let us denote the priority order for school s by $\succ_s$. Let us denote the preferences of student i by P_i. Let R_i denote the at-least-as-good-as relation associated with P_i.

A **student placement problem** is a pair $((\succ_s)_{s \in S}, (P_i)_{i \in I})$ consisting of a collection of priority orders and a preference profile.[1] For a given student placement problem, at an **allocation** each student is placed to only one school and the number of students placed to a particular school does not exceed the number of available seats at that school. An allocation is **Pareto efficient** if there is no other allocation which makes all students at least as well off and at least one student better off.

A very closely related problem to the student placement problem is the well-known *college admissions problem* due to Gale and Shapley (1962). The only but crucial difference between the two problems is that in a college admissions problem, schools have preferences over students whereas here, schools are merely objects (each of which has multiple copies) to be consumed. A central concept in college admissions is "stability." An allocation is **stable** if there is no student-school pair (i, s) such that student i prefers school s to the school he is placed to, and school s prefers student i to at least one student who is placed to it. The natural counterpart of stability in our context is "fairness." We say that at an allocation a student i has **justified envy** for a student at school s if he prefers school s to the school he is placed to, and he has higher priority for school s than at least one student who is placed to it. An allocation is **fair** if there is no student-school pair (i, s) such

[1] The student placement problem is also closely related to the "house allocation problem" in which there is a set of objects collectively owned by the society. See for example, Pápai (2000), Abdulkadiroğlu and Sönmez (1998, 1999), Ehlers et al. (2002), Ergin (2000), Ehlers and Klaus (2003), Ehlers (2002a), and Kesten (2003a,b,c).

that student i has justified envy for a student who is placed to school s.

A **student placement mechanism** or, simply a mechanism, selects an allocation for each student placement problem. A student placement mechanism is Pareto efficient if it always selects Pareto efficient allocations. A student placement mechanism is fair if it always selects fair allocations. A student placement mechanism is **strategy-proof**[2] if no student can ever gain by misstating his preferences.

Student placement is an important real-life problem. We first examine the mechanisms that are currently in use as well as those that have been proposed in the existing mechanism design literature. We start with a frequently-used real-life student placement mechanism.

2.1 Boston Student Placement Mechanism

We first examine a popular real-life mechanism: the **Boston mechanism**. This mechanism and its slight variants are currently being used in many places such as Boston, Seattle, Minneapolis, Lee County, and Florida. For a given student placement problem, the outcome of the Boston mechanism can be found via the following algorithm:

Step 1: Only the first choices of the students are considered. For each school, consider only those students who have listed it as their first choice and place these students to this school one at a time following their priority order until either there are no seats left or there is no student who has listed it as his first choice, is left.

Step 2: Consider the remaining students. Only the second choices of these students are considered. For each school with still available seats, consider only those students who have listed it as their second choice and place these students to this school one at a time following their priority order until either there are no seats left or there is no student who has listed it as his second choice, is left.

In general,

Step k, $k \geq 2$: Consider the remaining students. Only the k-th choices of these students are considered. For each school with still available seats, consider only those students who have listed it as their k-th choice and place

[2] This requirement is also referred as *incentive compatibility*.

these students to this school one at a time following their priority order until either there are no seats left or there is no student who has listed it as his k-th choice, is left.

The Boston mechanism has serious deficiencies. First of all, it is not strategy-proof. More importantly, it gives very strong incentives to students for misstating their preferences. Because a student who has high priority for a school may lose his advantage for that school if he does not list that school as his best choice, this mechanism forces students to think very strategically. Moreover, even the school district authorities explicitly advice students and their parents to make strategic choices (see for example, Glazerman and Meyer (1996) pp. 26-27).

Many students in fear of losing their priority for certain schools, tend to submit preferences that do not represent their true choices. In a recent experiment on the Boston mechanism, Chen and Sönmez (2003) show that 80% of the subjects chose to misstate their preferences under the Boston mechanism. Misstating one's preferences receives so much promotion that even suggestions of the following kind appear in the press (originally quoted by Ergin and Sönmez, 2003):

> "Make a realistic, informed selection on the school you list as your first choice. It is the cleanest shot you will get at a school. But if you aim too high, you might miss.
>
> Here is why: If the random computer selection rejects your first choice, your chances of getting your second choice school are greatly diminished. That's because you fall in line behind everyone who wanted your school as their first choice. You can fall even farther back in line as you get bumped to your third, fourth, and fifth choices."
>
> *St. Petersburg Times* (September 14, 2003)

Ergin and Sönmez (2003) consider the preference revelation game induced by the Boston mechanism and show that the set of Nash equilibria outcomes of this game is equal to the set of stable allocations under the true preferences. Based on this result, they suggest the use of the student optimal stable mechanism (to be introduced next).

Finally, even if this mechanism is Pareto efficient with respect to the revealed preferences, since most students are not submitting their true preferences, it is very unlikely that the outcome is Pareto efficient (with respect to true preferences). As for equity considerations, obviously the Boston mechanism leads to serious violations of students' priorities. The next two mechanisms we describe are the ones that have previously been discussed in the literature.

2.2 Gale-Shapley Student Optimal Stable Mechanism

One of the two mechanisms Abdulkadiroğlu and Sönmez (2003) advocate is the **student optimal stable mechanism (SOSM)** due to Gale and Shapley (1962). For a given student placement problem, the outcome of this mechanism can be found via the following **deferred acceptance (DA)** algorithm:

Step 1: Each student applies to his first choice. For each school, consider only those students who have applied to it and tentatively place these students to this school one at a time following their priority order until there are no seats or students left. Any remaining students are rejected.

In general,

Step k, $k \geq 2$*: Each student who was rejected in the previous step applies to his next choice. For each school, consider only those students who have applied to it at this step and those who have been tentatively placed to it school at previous steps. Tentatively place these students to this school one at a time following their priority order until there are no seats or students left. Any remaining students are rejected.*

The algorithm terminates when no student is rejected any more. At termination, all tentative placements are finalized. For a given student placement problem, the DA algorithm chooses the favorite fair allocation of each student.

PROPOSITION 1 (Balinski and Sönmez, 1999) *The student optimal stable mechanism Pareto dominates*[3] *any other fair mechanism.*

[3] A mechanism φ *Pareto dominates* another mechanism ψ if for each student placement problem, each student finds his placement under φ at least as good as the one under ψ

Another important feature of SOSM is that no student can ever gain by misstating his preferences.

PROPOSITION 2 (Dubins and Freedman, 1981; Roth, 1982) *The student optimal stable mechanism is strategy-proof.*

However, SOSM has an important drawback. Its outcome is not necessarily Pareto efficient.[4] The following example illustrates this point.

Example 1 (The student optimal stable mechanism is not Pareto efficient): Let $I \equiv \{i_1, i_2, i_3\}$ and $S \equiv \{s_1, s_2, s_3\}$ where each school has only one seat. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_3	i_2	$\vdots$
i_1	i_3	
i_2		

P_{i_1}	P_{i_2}	P_{i_3}
s_1	s_1	s_2
$\underline{s_3}$	$\underline{s_2}$	$\underline{s_1}$
s_2	s_3	s_3

To calculate the outcome of the student optimal stable mechanism, we run the DA algorithm:

Step 1: Students i_1 and i_2 apply to school s_1 and student i_3 applies to school s_2. Because student i_1 has higher priority for school s_1 than student i_2, student i_1 is tentatively placed to school s_1 and student i_2 is rejected. Student i_3 is tentatively placed to school s_2.

Step 2: Student i_2 applies to his next choice, which is school s_2. Because student i_2 has higher priority for school s_2 than student i_3, student i_3 is rejected, and student i_2 is tentatively placed to school s_2.

Step 3: Student i_3 applies to his next choice, which is school s_1. Because student i_3 has higher priority for school s_1 than student i_1, student i_1 is rejected, and student i_3 is tentatively placed to school s_1.

Step 4: Student i_1 applies to his next choice, which is school s_3 from which he is not rejected. At this point the algorithm terminates.

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and there is at least one student placement problem at which (at least) one student finds his placement under φ better than the one under ψ.

[4]Ergin (2002) gives a sufficient and necessary condition on the collection of priority orders that ensures the Pareto efficiency of SOSM.

This algorithm is also summarized in the following table where at a given step, the students inside a rectangle are the ones who are tentatively occupying a seat at the corresponding school.

Step	s_1	s_2	s_3
1	$\boxed{i_1}$, i_2	$\boxed{i_3}$	
2	$\boxed{i_1}$	i_3, $\boxed{i_2}$	
3	i_1, $\boxed{i_3}$	$\boxed{i_2}$	
4	$\boxed{i_3}$	$\boxed{i_2}$	$\boxed{i_1}$

The outcome of the DA algorithm for this problem is the underlined allocation above. It is easy to see that this allocation is not Pareto efficient.

We will have more to say about the reason and the size of the efficiency loss for SOSM.

2.3 Top Trading Cycles Mechanism

The second mechanism Abdulkadiroğlu and Sönmez (2003) propose is what they call the **top trading cycles mechanism (TTCM)**. TTCM is based on *Gale's top trading cycles procedure* proposed in the context of "housing markets" (Shapley and Scarf, 1974). In a housing market, there is a set of "indivisible objects" (*e.g.*, houses) each of which is initially *assigned* to a different agent among a set of "agents." Gale's top trading cycles procedure works as follows:[5] Each agent points to the agent who is assigned his best choice object. Since the number of agents is finite, there is at least one cycle. Then in each cycle, the corresponding trades are performed (*i.e.,* each agent in a cycle is given the object the agent he points to is assigned), and these agents and objects are removed. Then the same procedure is applied to the new market and so on. The algorithm terminates when there are no agents left. This procedure yields an allocation which is in the *core* of the housing market.[6]

TTCM is one adaptation of Gale's top trading cycles procedure to the student placement context. Since now there may be multiple copies (seats)

[5] The procedure we describe here is not the same procedure proposed by Gale. But, the two are equivalent. For reasons to be obvious shortly, we adopt this alternative procedure.

[6] If preferences are strict, this allocation is unique (Roth and Postlewaite, 1977).

of a particular object (school), this difference in the models has to be taken into account. For a given student placement problem, the outcome of TTCM can be found via the following algorithm:[7]

Step 1: Each student who has the highest priority for a school is assigned all seats of that school. (A student may be assigned the seats of different schools.) Each student points to the student (possibly himself) who is assigned (all seats of) his best choice. There is at least one cycle. Each student in a cycle is placed to the school that was assigned to the student he is pointing to. Since each student who is part of a cycle is already placed to a school, he is removed and the number of available seats at that school is decreased by one.

Step 2: All the remaining seats of each school which were assigned to a student who was part of a cycle at Step 1 are assigned to the student with the highest priority for that school among the remaining students. (A student may be assigned the seats of different schools.) Each student points to the student (possibly himself) who is assigned (all remaining seats of) his best choice among the remaining schools. Each student in a cycle is placed to the school that was assigned to the student he is pointing to. Since each student who is part of a cycle is already placed to a school, he is removed and the number of available seats at that school is decreased by one.

In general,

Step k, $k \geq 2$*: All the remaining seats of each school which were assigned to a student who was part of a cycle at Step k-1 are assigned to the student with the highest priority for that school among the remaining students. (A student may be assigned the seats of different schools.) Each student points to the student (possibly himself) who is assigned (all remaining seats of) his best choice among the remaining schools. Each student in a cycle is placed to the school that was assigned to the student he is pointing to. Since each student who is part of a cycle is already placed to a school, he is*

[7]The algorithm we give here is an equivalent algorithm to the one that was proposed by Abdulkadiroğlu and Sönmez (2003). Since this version of the algorithm will make it easier for us to compare it with the alternative adoptation of Gale's top trading cycles procedure that we will propose later in the paper, we give this equivalent algorithm. This alternative algorithm is proposed in Kesten (2003a).

removed and the number of available seats at that school is decreased by one.

The algorithm terminates when no student is left.

Being based on Gale's top trading cycles procedure, TTCM inherits the desirable properties it has. The first one such property is Pareto efficiency.

PROPOSITION 3 (Abdulkadiroğlu and Sönmez, 2003) *The top trading cycles mechanism is Pareto efficient.*

However, TTCM is not fair.[8] (We will shortly return to this aspect of TTCM.) The second important property TTCM has is strategy-proofness.

PROPOSITION 4 (Abdulkadiroğlu and Sönmez, 2003) *The top trading cycles mechanism is strategy-proof.*

2.4 Pareto Efficiency and Fairness: An Impossibility

A well-known negative result for the student placement context (as well as for the two-sided matching) is the incompatibility between Pareto efficiency and fairness.

PROPOSITION 5 (Roth 1982) *A Pareto efficient and fair allocation may not always exist and if it exists, it is unique.*

By Proposition 1, if a Pareto efficient and fair allocation exists for a given student placement problem, then it is the one selected by the student optimal stable mechanism. Proposition 5 has two immediate corollaries:

COROLLARY 1 *There is no mechanism that is Pareto efficient and fair.*

COROLLARY 2 *The student optimal stable mechanism is not Pareto efficient and the top trading cycles mechanism is not fair.*

[8] Recently, Kesten (2003a) gives a sufficient and necessary condition (which is similar to the one given in Ergin, 2002) for the equivalence of SOSM and TTCM or, the fairness of TTCM, or the resource monotonicity of TTCM, or the population monotonicity of TTCM.

3 A New Equity Criterion: Reasonable Fairness

3.1 Motivation for a new equity criterion

It is clear that "fairness" is the most natural and ideal equity criterion in this context. However, by Corollary 1, there is no mechanism that is both fair and Pareto efficient. If one values fairness over Pareto efficiency, then the student optimal stable mechanism is, of course, the obvious choice (recall Proposition 1). However, because we care about equity does not mean that we can totally ignore the welfare aspects of the problem at hand. Better yet, it becomes completely pointless to insist on such a criterion, if it causes every student to be unsatisfied. We know that the student optimal stable mechanism is not Pareto efficient. However, the following example shows that the size of the efficiency loss due to this mechanism can be disappointingly large.

Example 2 (The student optimal stable mechanism may result in severe efficiency loss): Let $I \equiv \{i_1, i_2, \ldots, i_{12}\}$ and $S \equiv \{s_1, s_2, \ldots, s_5\}$ where each school except s_5 has two seats and school s_5 has four seats. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
i_5	i_2	i_{11}	i_4	i_{12}
i_7	i_{10}	i_1	i_8	i_9
i_{12}	i_9	i_6	i_3	i_6
i_4	i_7	i_{10}	i_1	i_3
i_8	i_4	i_5	i_2	$\vdots$
i_9	i_3	i_{12}	i_6	
i_{10}	i_1	i_4	i_7	
i_{11}	i_8	i_2	i_5	
i_6	i_{12}	i_3	i_9	
i_1	i_4	i_7	i_{10}	
i_2	i_5	i_8	i_{11}	
i_3	i_6	i_9	i_{12}	

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	P_{i_7}	P_{i_8}	P_{i_9}	$P_{i_{10}}$	$P_{i_{11}}$	$P_{i_{12}}$
$\boxed{s_1}$	$\boxed{s_1}$	$\boxed{s_1}$	$\boxed{s_2}$	$\boxed{s_2}$	$\boxed{s_2}$	$\boxed{s_3}$	$\boxed{s_3}$	s_3	s_4	s_4	s_4
s_2	s_3	s_3	s_3	s_4	s_1	s_4	s_2	s_4	s_1	s_1	s_2
s_4	s_4	s_2	s_1	s_3	s_4	s_2	s_1	s_1	s_3	s_2	s_3
$\underline{s_3}$	$\underline{s_2}$	s_4	$\underline{s_4}$	$\underline{s_1}$	s_3	$\underline{s_1}$	$\underline{s_4}$	s_2	$\boxed{\underline{s_2}}$	$\boxed{\underline{s_3}}$	s_1
s_5	s_5	$\underline{s_5}$	s_5	s_5	$\underline{s_5}$	s_5	s_5	$\boxed{\underline{s_5}}$	s_5	s_5	$\boxed{\underline{s_5}}$

The outcome of the DA algorithm for this student placement problem is the above underlined allocation. (See Appendix for details.) It is striking to see that the student optimal stable mechanism places each student to either his worst choice or his second worst choice. This allocation is clearly Pareto inefficient. For example, the allocation, marked above with rectangles, which places the first eight students to their best choices and that does not change the placement of the last four students, Pareto dominates the above allocation.

It is quite easy to construct an example like the one above for student placement problems of much bigger sizes. It is also noteworthy to observe that the size of the potential efficiency loss in the outcome of the student optimal stable mechanism increases as the number of students rejected from schools increases. (We will make this argument more precise in the next section.) Even though each student may not end up at his almost bottom choice at every student placement problem under the student optimal stable mechanism, a great number of students may suffer from this welfare loss. Taking the fact that it is the students' future at stake into account, it is clear that one can not take any chances about this issue.

Recall Corollaries 1 and 2. Since there may not always exist a Pareto efficient and fair allocation, clearly it is not possible to have a Pareto efficient and fair mechanism. The point we have to be careful about this result is that a mechanism can be excused for violating either property only when it is not possible to find any allocation satisfying both properties. However, it is very natural to require and expect a good mechanism to select the Pareto efficient and fair allocation whenever it exists. Note that the student optimal stable mechanism meets this requirement. However, the top trading cycles mechanism fails to do so.

Example 3 (The top trading cycles mechanism may not select the Pareto efficient and fair allocation, even if such an allocation exists): Let $I \equiv \{i_1, i_2, i_3\}$ and $S \equiv \{s_1, s_2\}$ where school s_1 has one seat and school s_2 has two seats. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$
i_1	i_3
i_2	i_2
i_3	i_1

P_{i_1}	P_{i_2}	P_{i_3}
$\boxed{s_2}$	$\underline{s_1}$	$\boxed{s_1}$
s_1	$\boxed{s_2}$	$\underline{s_2}$

It is easy to calculate that the outcome of the top trading cycles mechanism is the underlined allocation above and the outcome of the student optimal stable mechanism is the allocation marked with rectangles. Note that the unique Pareto efficient and fair allocation for this problem is the one given by the student optimal stable mechanism. However, the outcome of the top trading cycles mechanism is clearly not this one.

The observation we make from Example 3 is that the top trading cycles mechanism may result in unnecessary justified envy among students. Of course this need not be the case only when there exists a Pareto efficient and fair allocation. It may very well be the case in general. In the following lemma, we make an interesting observation about Pareto efficient mechanisms.

LEMMA 1 *Let φ be a Pareto efficient mechanism. Given a student placement problem, if a student prefers his placement under the student optimal stable mechanism to his placement under φ, then under φ he either has justified envy for the school to which he is placed under the student optimal stable mechanism or, there is another student who also prefers his placement under the student optimal stable mechanism to his placement under φ.*

Note that if φ is a Pareto efficient mechanism that selects a different allocation than the student optimal stable mechanism for a student placement problem, then by Proposition 5, there is at least one student who has justified envy for another student at the allocation selected by φ. Lemma 1 says that if the Pareto efficient mechanism φ places a student to a school worse for him than his placement under the student optimal stable mechanism, then the only way he does not have justified envy for another student who is placed to that school is that another student is also placed to a school worse for him than his placement under the student optimal stable mechanism. Similarly, the only way that student does not have justified envy for another student who is placed to that school is that, another student is also placed to a school worse for him than his placement under the student optimal stable mechanism and so on. What this means is that if a Pareto efficient mechanism that does not Pareto dominate the student optimal stable mechanism avoids situations in which students have justified envy for the schools they would be placed under the student optimal stable mechanism, then the cost it has to pay is to increase the number of students who prefer their placement under the student optimal stable mechanism. This, in turn, means that even though φ is Pareto efficient, the welfare distribution under φ would be quite unbalanced. In this sense, the student optimal stable mechanism outcome can be seen as a benchmark for a balanced trade-off between equity and optimality considerations. Next we examine the top trading cycles mechanism in terms of students' welfare as compared to the student optimal stable mechanism.

Example 4 (The top trading cycles mechanism outcome may not Pareto dominate the student optimal stable mechanism outcome, even if the student optimal stable mechanism outcome is Pareto inefficient): Let $I \equiv \{i_1, i_2, i_3, i_4\}$ and $S \equiv \{s_1, s_2, s_3, s_4\}$ where each school has only one seat. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
i_4	i_2	$\vdots$	i_1
i_3	i_3		$\vdots$
i_1	$\vdots$		
i_2			

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
$\boxed{s_1}$	s_1	s_2	$\underline{\boxed{s_4}}$
$\underline{s_3}$	$\underline{\boxed{s_2}}$	$\underline{s_1}$	$\vdots$
$\vdots$	$\vdots$	$\boxed{s_3}$	
		$\vdots$	

The outcome of the student optimal stable mechanism is the underlined allocation above. Note that this allocation is Pareto inefficient. (If the placements of students i_2 and i_3 are exchanged, we obtain a Pareto improvement.) The outcome of the top trading cycles mechanism is the allocation marked with rectangles. Clearly, student i_3 prefers his placement under the student optimal stable mechanism to his placement under the top trading cycles mechanism.

3.2 Reasonable Fairness

Equity and optimality are two very important considerations in the student placement context. Our not being able to satisfy these two properties to the full extent, by no means implies that we should totally disregard one of the two properties. A more plausible approach is that one can look for requirements that will involve both properties at a satisfactory degree. In this regard, we propose an alternative equity criterion that will attain a balanced trade-off between the two.

Consider an allocation at which there is a student who has justified envy for another student who is placed to a certain school. That is, there is a student who would rather be placed to another school for which he has higher priority than another student who is currently placed to that school. Naturally, such a student would consider this allocation as ∈unfair. Consequently, he would object to that allocation and demand a fair allocation to be selected. Note that when this student objects to the initial allocation, he hopes that at the new allocation, he will be placed to the school which he was not placed earlier or to an even better school for him. If there is indeed a fair allocation that makes this student better off as compared to the initial allocation, then the point this student makes is reasonable and he is indeed right in his objection. But, what if there is no such fair allocation? That is, what if there is no fair allocation that places this student to either the school to which he was not placed earlier or, to a better school for him. In such a case, there is no way this student could be made better off and, moreover, any fair allocation can make him even worse off. That is in this case, the student's objection is groundless. The alternative equity criterion we propose is based on this observation.

Definition: *An allocation is **reasonably fair** for a given student placement problem if whenever a student i has justified envy for another student who is placed to a school s, then there is no fair allocation that places student i to school s or to a school better for him than school s.*

By Proposition 1, an equivalent way to state the above definition is the following:

Alternative Definition: *An allocation is reasonably fair for a given student placement problem if it is either fair or, no student is placed to a school that is worse for him than the school he is placed to at the student optimal stable mechanism outcome.*

A mechanism is reasonably fair if it always selects reasonably fair allocations.

4 A Reasonably Fair and Pareto Efficient Mechanism

Let us have a closer look at Example 1 where there is a student placement problem for which the outcome of the student optimal stable mechanism is not Pareto efficient. There, at the first step, since student i_1 had higher priority for school s_1 than student i_2, student i_2 was rejected from school s_1. Having rejected from school s_1, student i_2, in the second step, applied to school s_2 and caused, this time, student i_3 to be rejected from school s_2. But this, in turn, caused student i_1 to be rejected from school s_2 since student i_3 next applied to school s_1 for which he has higher priority than student i_1. In short, when student i_1 caused student i_2 to be rejected from school s_1, he started a chain of rejections which ended back at school s_1 where it formed a full cycle, and caused himself to be rejected. Such a cycle resulted in loss of efficiency. Indeed, it is this sort of cycles that cause the student optimal stable mechanism to be Pareto inefficient.[9] Suppose school s_1 was removed from student i_1's preferences without affecting the relative ranking of the other schools in his preferences. Note that, when we re-run the DA algorithm replacing the preferences of student i_1 by his new preferences, there is no difference in the placement of student i_1. But, because the previously mentioned cycle now disappears, students i_2 and i_3 benefit from this change. Moreover, the new allocation is now Pareto efficient.

To sum up, what is going on here is that, by applying to school s_1, student i_1 "blocked" a settlement between students i_1 and i_2 without affecting his own placement and introduced inefficiency into the outcome. The idea behind the mechanism we are going to introduce next is based on the identification of

[9] Ergin (2002) gives a sufficient condition on the collection of priority orders that makes sure this kind of cycles never form.

students like student i_1 of this example and preventing them from blocking potential settlements between other students.

Given a student placement problem to which the DA algorithm is applied, let i be a student who is tentatively placed to a school s at a Step t and rejected from it at a Step t'. If there is at least one other student who is rejected from school s after Step $t-1$ and before Step t' (*i.e.*, rejected at a Step $l \in \{t, t+1, \ldots, t'-1\}$), then we call student i a **blocker** for school s and the pair (i, s) a **blocking pair**.

The mechanism we propose is based on the identification of blocking pairs in the DA algorithm and suitably taking care of them. Despite being seemingly straightforward, such a job becomes quite intrigue as the size of the problem increases. For example, for a given problem there may be more than one blockers for the same school and the rejection chains in the DA algorithm could have a nested and complicated structure making it difficult to identify which blocker (or blockers) is (are) the actual reason for the inefficiency. To illustrate, we give an example.

Example 5 (A student placement problem where the associated DA algorithm contains nested rejection chains): Let $I \equiv \{i_1, i_2, \ldots, i_6\}$ and $S \equiv \{s_1, s_2, \ldots, s_5\}$ where each school except s_5 has only one seat and school s_5 has two seats. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
i_2	i_3	i_1	i_4	$\vdots$
i_1	i_6	i_6	i_3	
i_5	i_4	i_2	i_6	
i_6	i_1	i_3	$\vdots$	
i_4	$\vdots$	$\vdots$		
i_3				

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
$\boxed{s_2}$	$\boxed{s_3}$	s_3	$\boxed{s_1}$	s_1	s_4
s_1	$\underline{s_1}$	$\boxed{s_4}$	s_2	$\underline{\boxed{s_5}}$	s_1
$\underline{s_3}$	$\vdots$	$\underline{s_2}$	$\underline{s_4}$	$\vdots$	s_3
$\vdots$		$\vdots$			s_2
					$\underline{\boxed{s_5}}$

The DA algorithm applied to this problem is summarized in the following table. To help us identify the rejection chains, if a student remains tentatively placed at a school for a certain number of steps, we use vertical dots to denote this. The SSOM outcome is the underlined allocation above.

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}$, i_4	$\boxed{i_1}$	$\boxed{i_2}$, i_3	$\boxed{i_6}$	
2	$\vdots$	i_1, $\boxed{i_4}$	$\vdots$	i_6, $\boxed{i_3}$	
3	i_5, i_6, $\boxed{i_1}$	$\vdots$		$\vdots$	
4	$\vdots$		i_2, $\boxed{i_6}$		$\boxed{i_5}$
5	$\boxed{i_2}$, i_1		$\vdots$		$\vdots$
6	$\vdots$		$\boxed{i_1}$, i_6		
7		$\boxed{i_6}$, i_4	$\vdots$		
8		$\vdots$		$\boxed{i_4}$, i_3	
9		i_6, $\boxed{i_3}$		$\vdots$	
10	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_5, i_6}$

Here, for example, there are two blockers for school s_1 : student i_5 (because student i_4 was rejected while he was tentatively at school s_1) and student i_1 (students i_5 and i_6 were rejected while he was tentatively at school s_1). Similarly, students i_4 and i_6 are blockers for school s_2, and students i_2 and i_6 are blockers for school s_3 *etc.* At this point an idea that comes into mind to take care of the efficiency loss is to remove from each blocker's preferences those schools he is a blocker for. However, that would not work, because we may end up having vacant seats at schools, although there are students who would rather be placed there. One can also easily check that doing the same exercise, this time, starting with those who have been blockers first, would not work either. However, when we do this exercise starting with the most recent blocker(s) and iteratively applying the removal procedure until no blockers are left, this gives us a unique outcome. We make this argument more precise in the following algorithm for a given student placement problem.

Round 0: Run the DA algorithm.

Round 1: Find the last step (of the DA algorithm ran in Round 0) at which a blocker is rejected from the school for which he is a blocker. Identify all the blocking pairs of that step. (If there are no blocking pairs, then stop.) For each identified blocking pair (i, s), remove school s from the preferences *of student i. Do not make any changes in the preferences of the remaining students. Re-run the DA algorithm with the new preference profile.*

In general,

Round k, $k \geq 2$: Find the last step (of the DA algorithm ran in Round k-1) at which a blocker is rejected from the school for which he is a blocker. Identify all the blocking pairs of that step. (If there are no blocking pairs, then stop.) For each identified blocking pair (i, s), remove school s from the preferences of student i. Do not make any changes in the preferences of the remaining students. Re-run the DA algorithm with the new preference profile.

Since the number of schools and students is finite, the algorithm eventually terminates. At termination, the outcome obtained at the final round is the outcome of the algorithm. We call the mechanism that associates to each student placement problem the outcome of the above algorithm the **efficiency adjusted deferred acceptance mechanism (EADAM)**. We next give an example.

Example 6: Let us find the outcome the efficiency adjusted deferred acceptance mechanism for the student placement problem given in Example 5.

Round 0: We run the DA algorithm. The table given in Example 5 shows the steps of the DA algorithm for this problem.

Round 1: The last step at which a blocker is rejected from the school he is a blocker for is Step 9 where the blocking pair is (i_6, s_2). (Student i_6 is a blocker for school s_2 because there is a student (namely, student i_4) who was rejected from school s_2 at the step student i_6 was tentatively placed to school s_3.) We remove school s_2 from the preferences of student i_6 and keep the preferences of the remaining students the same. We then re-run the DA algorithm with the new preference profile:

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}$, i_4	$\boxed{i_1}$	$\boxed{i_2}$, i_3	$\boxed{i_6}$	
2	$\vdots$	i_1, $\boxed{i_4}$	$\vdots$	i_6, $\boxed{i_3}$	
3	i_5, i_6, $\boxed{i_1}$	$\vdots$		$\vdots$	
4	$\vdots$		i_2, $\boxed{i_6}$		$\boxed{i_5}$
5	$\boxed{i_2}$, i_1		$\vdots$		$\vdots$
6	$\vdots$		$\boxed{i_1}$, i_6		
7	$\boxed{i_2}$	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_3}$	$\boxed{i_5, i_6}$

Round 2: The last step at which a blocker is rejected from the school he is a blocker for is Step 6 where the blocking pair is (i_6, s_3). We remove school s_3 from the (updated) preferences of student i_6 and keep the preferences of the remaining students the same. We then re-run the DA algorithm with the new preference profile:

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}$, i_4	$\boxed{i_1}$	$\boxed{i_2}$, i_3	$\boxed{i_6}$	
2	$\vdots$	i_1, $\boxed{i_4}$	$\vdots$	i_6, $\boxed{i_3}$	
3	i_5, i_6, $\boxed{i_1}$	$\vdots$		$\vdots$	
4	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_5, i_6}$

Round 3: The last step at which a blocker is rejected from the school he is a blocker for is Step 3 where the blocking pair is (i_5, s_1). We remove school s_1 from the preferences of student i_5 and keep the preferences of the remaining students the same. We then re-run the DA algorithm with the new preference profile:

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}$, i_3	$\boxed{i_6}$	$\boxed{i_5}$
2	$\vdots$	$\vdots$	$\vdots$	i_6, $\boxed{i_3}$	$\vdots$
3	$\boxed{i_6}$, i_4			$\vdots$	
4	$\vdots$	i_1, $\boxed{i_4}$	$\boxed{i_2}$	$\boxed{i_3}$	
5	i_6, $\boxed{i_1}$				
6	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_5, i_6}$

Round 4: The last step at which a blocker is rejected from the school he is a blocker for is Step 5 where the blocking pair is (i_6, s_1). We remove school s_1 from the (updated) preferences of student i_6 and keep the preferences of the remaining students the same. We then re-run the DA algorithm with the new preference profile:

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}$, i_3	$\boxed{i_6}$	$\boxed{i_5}$
2	$\vdots$	$\vdots$	$\vdots$	i_6, $\boxed{i_3}$	$\vdots$
3	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_5, i_6}$

Round 5: There are no blocking pairs, hence we stop. The outcome of the efficiency adjusted deferred acceptance mechanism is the allocation obtained at the end of Round 4. This allocation is marked with rectangles on the preference profile given in Example 5. The reader may easily check that this allocation is Pareto efficient.

EADAM very much mimics the student optimal stable mechanism. This allows it to inherit a great deal of the nice equity properties of the student optimal stable mechanism. At each round of the algorithm, no student is made worse off as compared to his placement in the previous round, but some students may be made better off. This continues until the procedure finally reaches a Pareto efficient allocation. This feature of the algorithm enables EADAM to achieve the equity criterion we have proposed in the previous section.

PROPOSITION 7 *The efficiency adjusted deferred acceptance mechanism is reasonably fair.*

COROLLARY 3 *The efficiency adjusted deferred acceptance mechanism Pareto dominates any fair mechanism. In particular, it Pareto dominates the student optimal stable mechanism.*

Another advantage of EADAM is that a student who has justified envy for another student under EADAM may be placed to an even worse school for him under the student optimal stable mechanism which is the best placement he can get under a fair mechanism.

Example 7 (A student who has justified envy for another student under the efficiency adjusted deferred acceptance mechanism may be placed to an even worse school for him under the student optimal stable mechanism): Let $I \equiv \{i_1, i_2, i_3, i_4, i_5\}$ and $S \equiv \{s_1, s_2, s_3, s_4, s_5\}$ where each school has only one seat. The priorities for schools and the preferences of students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
i_3	i_2	i_5	i_1	$\vdots$
i_1	i_3	i_4	i_2	
i_2	$\vdots$	i_1	$\vdots$	
$\vdots$		$\vdots$		

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}
s_1	$\boxed{s_1}$	$\boxed{s_2}$	s_3	$\boxed{s_4}$
$\boxed{s_3}$	$\underline{s_2}$	$\underline{s_1}$	$\underline{\boxed{s_5}}$	$\underline{s_5}$
$\underline{s_4}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$				

The DA algorithm applied to this problem is given in the following table.

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_1}$, i_2	$\boxed{i_3}$	$\boxed{i_4}$	$\boxed{i_5}$	
2	$\vdots$	i_3, $\boxed{i_2}$	$\vdots$	$\vdots$	
3	i_1, $\boxed{i_3}$	$\vdots$			
4	$\vdots$		i_1, $\boxed{i_4}$		
5			$\vdots$	i_5, $\boxed{i_1}$	
6			$\boxed{i_5}$, i_4		
7	$\boxed{i_3}$	$\boxed{i_2}$	$\boxed{i_5}$	$\boxed{i_1}$	$\boxed{i_4}$

The outcomes of SOSM (the underlined allocation) and EADAM (the allocation marked with rectangles) are shown above. Note that at the EADAM allocation, student i_1 has justified envy for student i_2 who is placed to school s_1. However, student i_1 is placed to school s_4 (a school worse for him than his placement under EADAM) under SOSM.

Noting the point made in Example 7 down as another plus of the efficiency adjusted deferred acceptance mechanism, we can easily say that this mechanism leaves almost no room to students for objection to the outcome. Next we state another important property of the efficiency adjusted deferred acceptance mechanism.

PROPOSITION 8 *The efficiency adjusted deferred acceptance mechanism is Pareto efficient.*

COROLLARY 4 *The efficiency adjusted deferred acceptance mechanism selects the fair and Pareto efficient allocation whenever it exists.*

4.1 Strategic Issues

Example 8 (The efficiency adjusted deferred acceptance mechanism is not strategy-proof): Let $I \equiv \{i_1, i_2, i_3\}$ and $S \equiv \{s_1, s_2, s_3\}$ where each school has only one seat. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_3	i_2	i_2
i_1	i_3	i_1
i_2		

P_{i_1}	P_{i_2}	P_{i_3}
$\underline{s_1}$	$\boxed{s_1}$	$\underline{\boxed{s_3}}$
$\boxed{s_2}$	$\underline{s_2}$	s_1
	s_3	

To calculate the outcome of EADAM, we first run the DA algorithm at Round 0:

Step 1: Students i_1 and i_2 apply to school s_1 and student i_3 applies to school s_3. Because student i_1 has higher priority for school s_1 than student i_2, student i_1 is tentatively placed to school s_1 and student i_2 is rejected. Student i_3 is also tentatively placed to school s_3.

Step 2: Student i_2 applies to his next choice, which is school s_2, and he is tentatively placed to this school.

Since the outcome of the DA algorithm is already Pareto efficient, we stop at the end of Round 0. The outcome of EADAM when all students truthfully report their preferences is given by the underlined allocation.

Now, suppose that student i_2 reports the fake preferences in which he exchanges the places of s_2 and s_3 in his true preferences. Let us re-calculate the outcome of EADAM. At Round 0, we run the DA algorithm:

Step1: This step is identical to Step 1 above: Student i_1 is tentatively placed to school s_1 and student i_3 to school s_3. Student i_2 is rejected.

Step 2: Student i_2 applies to his next choice, which is, this time, school s_3 and since he has higher priority for it than student i_3, he is tentatively placed to school s_3 and student i_3 is rejected.

Step 3: Student i_3 applies to his next choice which is school s_2. Since student i_3 has higher priority for this school than student i_1, he is tentatively placed to school s_3 and student i_1 is rejected.

Step 4: Student i_1 applies to his next choice which is school s_2 and since he is the first to apply to school s_2, he is tentatively placed to it. No student is rejected at the end of this step and the DA algorithm terminates.

Round 1: Since student i_1 is rejected from school s_1 at Step 3 and since student i_2 has been rejected from school s_1 while student i_1 was tentatively placed to school s_1, we identify (i_1, s_1) as the last and the only blocking pair. Next, we remove school s_1 from student i_1's preferences and re-run the DA algorithm with the new preference profile:

Step 1: Students i_1, i_2, and i_3 apply to schools s_2, s_1, and s_3 respectively. No student is rejected from any school and the DA algorithm terminates.

Round 2: There are no blocking pairs and we stop. The outcome of EADAM when student i_2 misstates his preferences is the allocation marked with rectangles above.

Note that student i_2 was able to manipulate EADAM by misstating his preferences. The reason this happened is that by switching the positions of two schools in his preferences, student i_2 initiated a rejection chain which eventually caused student i_1 to be rejected from school s_1. This, in turn, caused us to identify (i_1, s_1) as a blocking pair and take student i_1's right to apply to school s_1 from him, which thus has benefited student i_2.

The following proposition shows that the vulnerability to strategic behavior can not be avoided by any reasonably fair and Pareto efficient mechanism.

PROPOSITION 9 *There is no mechanism that is reasonably fair, Pareto efficient, and strategy-proof.*

One point here is very critical: for a mechanism, being not strategy-proof is not the same thing as being easily manipulable. A mechanism may not be strategy-proof but it may still be very difficult to manipulate or even to identify a clear strategy that would allow a student to successfully manipulate the mechanism. This point is very crucial in terms of the practical applicability of a mechanism. For example, let us consider the Boston mechanism again. A student who gives very high ranking in his true preferences to certain schools for which he has low priority can easily identify it as an almost 100% right strategy not to list those schools at the top positions in the preferences he submits (unless of course, he has some extra information about those schools). As we have argued in the relevant section, even the press gives clear guidelines on how students should misstate their preferences.

Now let us compare this case with the case of EADAM. Example 8 is probably one of the simplest examples to illustrate how EADAM can be manipulated. There, in order for student i_2 to gain by misstating his preferences, he had to switch the ranking of two schools on his preferences. To manipulate, he submits preferences in which he increased the ranking of a school (school s_2) and decreased the ranking of an actually better school for him (school s_3). By doing this, he initiated a rejection chain which caused a student i_1 to be rejected from school s_1, in turn, resulting in student i_1 being identified as a blocker for school s_1 the school student i_2 eventually got himself placed to. First of all, if a student ever attempts this kind of a manipulation, he has to know other student's preferences as well as the priority orders for schools, an information no student (or parent) can obtain in real-life. Second of all, the complexity of the calculations such a student needs to make is inarguably very high. Finally and most importantly, in order to be able to successfully manipulate EADAM, a student has to switch the positions of two (or more) schools in his true ranking. What this means is that he has to take the risk of ending up at that worse school for him which he reported as a better school as compared to his true preferences.

Mechanisms based on the DA algorithm are also used in various real-life applications such as the US intern-hospital market to match interns to hospitals (see, for example, Roth (1984), Roth and Peranson (1988, 1999), and Roth and Rothblum (1999)). For example, the hospital optimal stable mechanism (which uses the DA algorithm in which hospitals propose to interns) is not strategy-proof. The way this mechanism can be manipulated by interns is based on exactly the same idea how EADAM can be manipulated: An intern by misstating his true preferences initiates a rejection chain which eventually causes a more preferable (than his assignment when he submits his true preferences) hospital to propose to him. Even though this DA based mechanism is not strategy-proof, in the last two decades all those markets in which the DA algorithm (or, its variants) are used have operated very successfully as opposed to the markets that use alternative procedures. Roth and Rothblum

(1999) search for the reason for this success and argue that this may be due to the fact that interns have "limited information" about the preferences of other interns because in such a case the scope of potentially profitable strategic behavior is considerably reduced. They show that regardless of the attitude of an intern toward risk, in a low information environment, it is never profitable for an intern to simply switch the positions of two hospitals in his preference ranking.[10] Although the manipulating side of the market there, is the non-proposing side in the DA algorithm, the way a manipulation occurs is identical and the entire intuition behind the observation made in Roth and Rothblum (1999) carries over to our context.

Let us make the argument more concrete. Very often an intern is not able to distinguish between two hospitals, and is not sure how other interns rank those two hospitals. That is, he sees them to be *symmetric*. Then this symmetry is captured in his beliefs. We say a student's information is symmetric for two schools s and s', if given that his own preferences are fixed, he assigns the same probability to any student placement problem and to its symmetric problem where the roles of s and s' are exchanged (see Appendix B for a formal treatment).

Recently, Ehlers (2002b) gives two conditions that are sufficient for a mechanism to be immune to strategic behavior due to a switch of two alternatives in the preferences of an intern (here, student) when he has limited information about other interns. These are "neutrality"[11] and "positive association." Neutrality requires that the mechanism should treat all schools equally. That is, the names of schools should not matter. This requirement is clearly satisfied by EADAM. His second condition is "positive association." It requires that given a student i who is placed to a school s when he submits the preferences P_i, if he exchanges the positions of school s and a school s' which he prefers to school s, then he is still placed to school s. EADAM does satisfy this property, too (see Appendix B). Therefore, in a limited information environment, it is never beneficial for a student to switch the true ranking of two schools in the preferences he states.

Let us interpret the stated preferences of a student as his strategy. Given a student placement problem $((\succ_s)_{s\in S}, (P_i)_{i\in I})$, a student i and two preferences P_i' and P_i'', we say strategy P_i stochastically dominates strategy P_i'

[10] See Theorem 1 of Roth and Rothblum (1999).

[11] Ehlers (2002b) actually calls this requirement "anonymity." Since, in our context, schools are merely objects, it would be more suitable to call this property neutrality.

if the probability distribution induced on the placements of student i when he states P_i', stochastically dominates the probability distribution induced on the placements of him when he states P_i'' where the comparison is made according to his true preferences (see Appendix B). Then using Theorem 1 of Ehlers (2002b), we obtain the following critical observation about EADAM.

PROPOSITION 10 *Under the efficiency adjusted deferred acceptance mechanism, any strategy which reverses the true ranking of two schools s and s' is stochastically dominated by a strategy which preserves the true ranking of s and s' for a student whose information for s and s' is symmetric.*

Proposition 10 has crucial importance in terms of the practical applicability of EADAM. It strongly discourages students from attempting to manipulate this mechanism. It is very unlikely that such an attempt would benefit any student, even if it is not totally impossible.

4.2 Efficiency adjusted Gale-Shapley mechanism with consent

At the allocation selected by EADAM, if there is a student i who has justified envy for some other student who is placed to a school s, then student i is a blocker for school s at a certain round of the algorithm (because this is only possible if that school has been removed from his preferences). Recall that when a student is identified as a blocker of a certain round, then this student is prevented from applying to this school by removing that school from his preferences. As a result of this, at the final allocation such a student may have justified envy for another student who is placed to that school. In fact, this is the only reason for having justified envy among some students under EADAM.[12]

One alternative way to perform the updating exercise on the preferences of students is to ask for the consent of the blocker student (because after all, what is being violated here is his priority for that school). Recall from Example 7 that while it does not make any blocker student worse off to grant

[12]It is not necessarily true, however, that at the EADAM allocation, every blocker has justified envy for some student who is placed to the school he is a blocker for. It is only true that if a student has justified envy for a student who is placed to a school at the EADAM allocation, then he must have been identified as a blocker for that school at some round.

such a permission, it may allow him to end up at a better school for himself by participating in this exercise. An important aspect of the algorithm we propose is that even if a blocker student does not grant permission for the updating exercise, it may still be possible to achieve a Pareto improvement on the SOSM outcome by performing the updating exercise with only those students who accept to conform.

If one would like to modify the algorithm to accommodate such a change, then this can be done in a straightforward way as follows: Again, start with the blocker(s) of the last step of the DA algorithm. If some blocker(s) does (do) not grant permission for the change in his (their) preferences, then proceed only with those who do. If there is only one blocker at a particular round and he does not grant permission, then find the last step at which a blocker can be identified and continue with those blockers in a similar way. Even if such a change in the algorithm may not allow us to achieve full Pareto efficiency[13], we may still be able to obtain a considerable Pareto improvement on the DA algorithm outcome.

To illustrate, for example, consider Example 7. We identify two blocking pairs during the entire algorithm. The student-school pair (i_4, s_3) is the blocking pair of Round 1 and (i_1, s_1) is the blocking pair of Round 2. Now, suppose student i_4 does not grant permission for school s_3 to be removed from his preferences. In such a case, we search for the next blocking pair which is (i_1, s_1). Suppose student i_1 grants permission for school s_1 to be removed from his preferences. Note that the allocation we obtain now is not Pareto efficient but it still Pareto dominates the student optimal stable mechanism outcome.[14]

5 A More Equitable Top Trading Cycles Mechanism

Abdulkadiroğlu and Sönmez (2003) have adopted *Gale's top trading cycles procedure* to the student placement context and proposed a Pareto efficient and strategy-proof mechanism. Despite its appealing properties, this mech-

[13] It may, too. Examples in which Pareto efficiency can still be achieved even if some blockers do not grant permission, are available from the author upon request.

[14] Because students i_2 and i_3 are now placed to schools s_1 and s_2 respectively, while everbody else is placed to the same school as he was placed under the student optimal stable mechanism.

anism is still open to criticism because of its equity aspects. As an example, suppose a school has five available seats. Then what this means is that each one of those five students who have the highest five priorities for this school, is entitled to one seat at this school. In this sense, given only this information, no student in these five has any superiority over any other student in the group for any other school. However, at the first step of the algorithm all these five seats are assigned to the student with the highest priority for that school. Since any student who has this school as his best choice has to point to this student, such a student is given all the trading rights of the seats of this school before any other student in the group of five which may potentially result in justified envy of students who have lower priority for that school but higher priority for the school this student is placed to. For example, consider Example 3. There, at the first step, student i_3 is assigned the two seats of school s_2 and student i_1 one seat of school s_1. Following our reasoning earlier, although student i_2 is also entitled to one seat at school s_2, since student i_3 is assigned all the seats of this school, student i_1 is forced to a trade with student i_3, which in turn leads to the violation the priority of student i_2 for school s_1. However, had student i_1 traded his right for one seat at school s_1 with student i_2 for his right for one seat at school s_2, there would not be any justified envy. Of course, due to the incompatibility between fairness and Pareto inefficiency, it can not be possible to totally avoid cases of justified envy. However, it may still be possible to remarkably decrease the number of these cases by considering an alternative adaptation of Gale's top trading cycles procedure.

In our approach, instead of assigning all the available seats of a school to the student with the highest priority for that school, we assign one seat to students following their priority order for that school until no seats are left. Also, instead of having each student point to the student who has the highest priority for his best choice, we let each student point to that student (among those who are assigned one seat from his best choice) who has the highest priority for the school he himself is assigned. (Note that in this case the student who is being pointed to is not an arbitrary person.) This ensures that the student who is being pointed to achieves a more advantageous position as compared to other students who are also assigned one seat from the same school, not because of the seat he is assigned but because of his high priority for the school whose seat the pointing student is assigned. This means, for example, if a cycle consisting of two students forms, then each student in that cycle has the highest priority for the school he is placed to within a certain group of people who are competing for the seats of those schools. Furthermore, by imposing priority restrictions on the rules students can point to one another, a great deal of potential for justified envy is eliminated, even if it may not be totally eliminated. To make this idea well-defined, we need to be more precise in the way the procedure is being carried out.

Here is a summary of our algorithm: At the first step, for each school seats are assigned to students one by one following their priority order to form student-seat pairs. A student can be contained in more than one student student-seat pair. We denote a student-seat pair by (i, s) where i is a student and, with a slight abuse of notation, s denotes one seat from school s. Each student-seat pair (i, s) points to the student-seat pair (i', s') such that (i) school s' is the best choice of student i and, (ii) student i' is the student with the highest priority for school s among the students who are assigned a seat from school s'. If there is already a student-seat pair at which student i is assigned one seat from his best choice school, then all student-seat pairs containing him point to that student-seat pair. Since the number of student-seat pairs is finite, there is at least one cycle. In each cycle, corresponding trades are performed, *i.e.*, if a student-seat pair (i, s) is pointing to the pair (i', s') in a cycle, then student i is placed to school s' and he is removed as well as the seat student i' is assigned.

Note that it is possible that some student-seat pairs which contain the same student appear in the same cycle or in different cycles. In such a case, that student is placed to his best choice and the extra seats of that school (for which other student-seat pairs containing him are pointing to in other cycles) remain to be *inherited* by the remaining students.[15] Also, if a student is removed and there are student-seat pairs containing him which do not participate in a cycle, then the seats assigned to him in those student-seat pairs also remain to be inherited. To sum up, seats remain to be inherited in two ways: (1) More than one student-seat pair containing the same student participate in a cycle or cycles: Then that student is given one seat from his best choice (*i.e.*, he is placed to his best choice) and the other seats of his best choice (for which he is pointing to in other cycles) remain to be inherited; (2) A student-seat pair participates in a cycle and there are other student-seat pairs containing the same student which do not participate in a

[15] Pápai (2000) introduces and characterizes quite a large family of rules which she calls "endowment inheritance rules" that are also based on Gale's top trading cycles algorithm. The idea of "inheritence of seats" we use here is inspired by that result.

cycle: Then the seats assigned to him in those student-seat pairs remain to be inherited. Once the corresponding trades for cycles of the first step are carried out, we move to the second step.

Those seats that remain to be inherited at the end of a Step t, $t \geq 1$, are not necessarily inherited at the very next step by the remaining students. Inheritance of seats of a school s does not take place until no student who was contained in a student-seat pair with a seat from school s at Step t is left.[16] Right after the step at the end of which the last student who was contained in a student-seat pair with a seat from school s at Step t is removed, all seats of school s which thus far remained to be inherited, are inherited by the remaining students one by one following their priority order, *i.e.*, these students are assigned those seats to again form student-seat pairs. At each step, again student-seat pairs point to each other in the way described above. Corresponding trades are carried out in each cycle and some seats remain to be inherited at the appropriate step. The procedure continues in a similar way. The following algorithm describes this procedure for a given student placement problem:

Step 1: For each school, all available seats are assigned to students one by one following their priority order to form student-seat pairs. Each student-seat pair (i, s) points to the student-seat pair (i', s') such that (i) school s' is the best choice of student i and, (ii) student i' is the student with the highest priority for school s among the students who are assigned a seat from school s'. If student i is already assigned one seat from his best choice school, then all student-seat pairs containing him point to that student-seat pair. There is at least one cycle. In each cycle, corresponding trades are performed and all student-seat pairs which participate in a cycle are removed. It is possible that the student-seat pairs containing the same student, say student i, appear in the same cycle or in different cycles. In such a case, student i is placed to his best choice and the other seats of that school (for which the student-seat pairs containing him are pointing to in those other cycles) remain to be inherited. For each student-seat pair (i, s) which participates in a cycle, the seats assigned to student i in other student-seat pairs which do not participate in a cycle also remain to be inherited.

Step 2: For each school s such that (i) there are seats of school s which

[16] As it will be clear shortly, this restriction ensures that our mechanism is strategy-proof.

remained to be inherited from Step 1 and, (ii) no student who was assigned a seat of school s at the first step is left, its seats which remained to be inherited are assigned to the remaining students one by one following their priority order to form new student-school pairs. Each student-seat pair (i, s) points to the student-seat pair (i', s') such that (i) school s' is the best choice of student i and, (ii) student i' is the student with the highest priority for school s among the students who are assigned a seat from school s'. If student i is already assigned one seat from his best choice school, then all student-seat pairs containing him point to that student-seat pair. There is at least one cycle. In each cycle, corresponding trades are performed and all student-seat pairs which participate in a cycle are removed. It is possible that the student-seat pairs containing the same student, say student i, appear in the same cycle or in different cycles. In such a case, student i is placed to his best choice and the other seats of that school (for which the student-seat pairs containing him are pointing to in those other cycles) remain to be inherited. For each student-seat pair (i, s) which participates in a cycle, the seats assigned to student i in other student-seat pairs which do not participate in a cycle also remain to be inherited.

In general,

Step k, $k \geq 2$: For each school s such that (i) there are seats of school s which remained to be inherited from previous steps , and (ii) no student who was assigned a seat of school s at a previous step is left, its seats which remained to be inherited from previous steps are assigned to the remaining students one by one following their priority order to form new student-school pairs. Each student-seat pair (i, s) points to the student-seat pair (i', s') such that (i) school s' is the best choice of student i and, (ii) student i' is the student with the highest priority for school s among the students who are assigned a seat from school s'. If student i is already assigned one seat from his best choice school, then all student-seat pairs containing him point to that student-seat pair. There is at least one cycle. In each cycle, corresponding trades are performed and all student-seat pairs which participate in a cycle are removed. It is possible that the student-seat pairs containing the same student, say student i, appear in the same cycle or in different cycles. In such a case, student i is placed to his best choice and the other seats of that school (for which the student-seat pairs containing him are pointing *to in those other cycles) remain to be inherited. For each student-seat pair (i, s) which participates in a cycle, the seats assigned to student i in other student-seat pairs which do not participate in a cycle also remain to be inherited.*

We call the mechanism that associates to each student placement problem the outcome of the above algorithm the **equitable top trading cycles mechanism (ETTCM)**. Next we give an example to illustrate the dynamics of this algorithm:

Example 9: Let $I \equiv \{i_1, i_2, \ldots, i_9\}$ and $S \equiv \{s_1, s_2, \ldots, s_5\}$ where schools s_1, s_3, and s_4 have three seats, schools s_2 and s_5 have two seats, and school s_4 has four seats. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
i_1	i_4	i_3	i_7	i_3
i_4	i_5	i_2	i_1	i_2
i_5	i_2	i_6	i_5	i_1
i_8	i_3	i_5	i_4	i_4
i_9	i_6	i_7	i_3	i_9
i_6	i_9	i_4	i_9	i_8
i_3	i_8	i_9	i_8	i_5
$\vdots$	$\vdots$	i_8	i_2	$\vdots$
		i_1	i_6	
		$\vdots$		

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	P_{i_7}	P_{i_8}	P_{i_9}
$\boxed{s_2}$	$\boxed{s_5}$	$\boxed{s_1}$	$\boxed{s_3}$	$\boxed{s_4}$	$\boxed{s_5}$	$\boxed{s_2}$	s_2	$\boxed{s_4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_4	s_1	s_5	$\vdots$
					$\vdots$	$\vdots$	$\boxed{s_3}$	

Step 1: Since students i_1, i_4, and i_5 have the first three highest priorities for school s_1, the three seats of this school are assigned to them to form the student-seat pairs (i_1, s_1), (i_4, s_1), and (i_5, s_1). Since students i_4 and i_5 have the first two highest priorities for school s_1, the two seats of this school are assigned to them to form the student-seat pairs (i_4, s_2) and (i_5, s_2). The other student-seat pairs which form similarly are: (i_3, s_3), (i_2, s_3), (i_6, s_3); (i_7, s_4), (i_1, s_4), (i_4, s_4); (i_3, s_5), and (i_2, s_5). Next we determine which student-seat pair points to which student-seat pair. Consider, for example, the student-seat pair (i_1, s_1). Since the best choice of student i_1 is school s_2, and each student-seat pair containing student i_1 will point to "one" of the student-seat pairs that contain one seat from school s_2, we first identify these student-seat pairs. They are (i_4, s_2) and (i_5, s_2). Since student i_4 has higher priority for

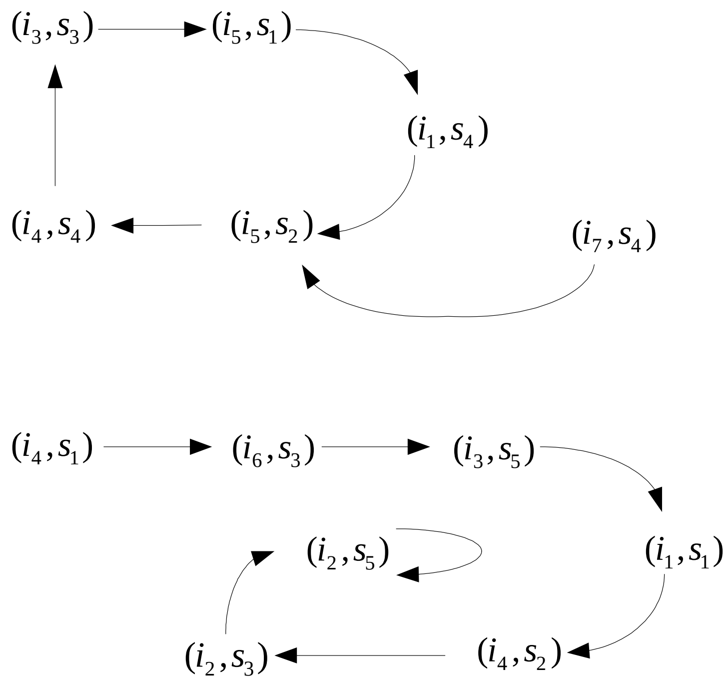

Figure 1: Step 1 of the ETTCM algorithm

the school student i_1 is assigned (namely, school s_1) at the student-seat pair (i_1, s_1), student-seat pair (i_1, s_1) points to the student-seat pair (i_4, s_2). Since there is a student-seat pair at which student i_2 is assigned his best choice, school s_5, all student-seat pairs containing student i_2 point to the student-seat pair (i_2, s_5). Other student-seat pairs point to each other in a similar way (see Figure 1).

Next we identify the cycles of Step 1. Note that two cycles form in this step: $\{(i_3, s_3), (i_5, s_1), (i_1, s_4), (i_5, s_2), (i_4, s_4)\}$ and a self-cycle $\{(i_2, s_5)\}$. Each student in a cycle is placed to the school whose seat is contained in the student-seat pair that the student-seat pair containing himself is pointing to. Thus, due to the first cycle, student i_3 is placed to school s_1, student i_5 to school s_4, student i_1 to school s_2, student i_5 to school s_4, and student i_4 to school s_3. Due to the second cycle, student i_2 is placed to school s_5.

All student-seat pairs which participate in a cycle are removed. Note that there are two student-seat pairs which both contain student i_5 and they both participate in a cycle. These are student-seat pairs (i_5, s_1) and (i_5, s_2). Each

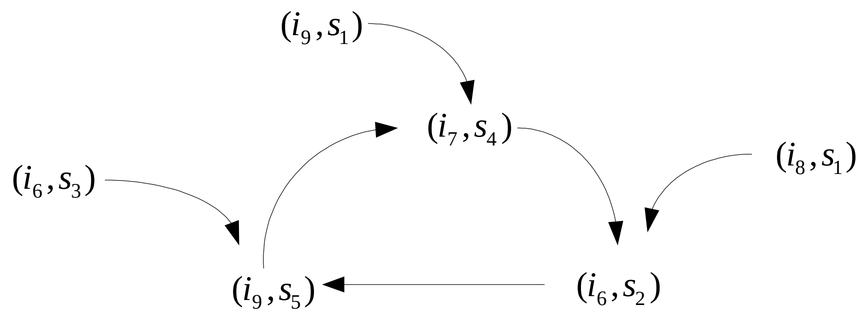

Figure 2: Step 2 of the ETTCM algorithm

of the two student-seat pairs are pointing to a student-seat pair containing the best choice of student i_5, namely school s_4. Then one of the two seats of school s_4 remain to be inherited. Also, since students i_1, i_2, i_3, and i_4 are removed and the student-seat pairs (i_1, s_1), (i_1, s_1), (i_2, s_3), (i_3, s_5), and (i_4, s_2) containing these students respectively, do not participate in a cycle, two seats of school s_1 and one seat of each of schools s_2, s_3, and s_5 also remain to be inherited.

Step 2: The remaining student-seat pairs from Step 1 are (i_6, s_3) and (i_7, s_4). Also, there are two seats of school s_1 and one seat of each of schools s_2, s_3, s_4, and s_5 to be inherited. Since there are still a student who is assigned one seat of school s_3 (namely student i_6) and a student who is assigned one seat of school s_4 (namely student i_7) at Step 1, there is no inheritance of the seats of schools s_3 and s_4 at this step. Since there is no student who is assigned a seat of schools s_1, s_2, and s_5 at Step 1 is left, two seats of school s_1 and one seat of each of s_2 and s_5 are inherited by the remaining students who have the highest priority for these school. More specifically, each one of students i_8 and i_9 inherits one seat from school s_1 and forms a student-seat pair with the seat he inherits. Also, student i_9 and student i_6 inherit one seat of school s_5 and school s_2 respectively, and they each form a student-seat pair with the seat he inherits. Thus, four new student-seat pairs form at Step 2: (i_8, s_1), (i_9, s_1), (i_9, s_5), and (i_6, s_2). Then student-seat pairs point to one another in the way described earlier (see Figure 2). There is only one cycle: $\{(i_6, s_2), (i_9, s_5), (i_7, s_4)\}$. Student i_6 is placed to school s_4, student i_9 to school s_4, and student i_7 to school s_2. These three student-seat pairs are removed. Since students i_6 and i_9 are removed and the student-seat pairs

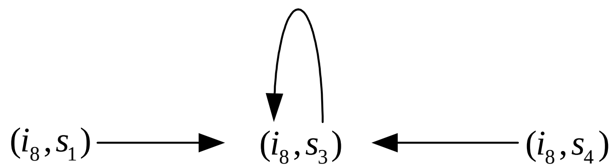

Figure 3: Step 3 of the ETTCM algorithm

(i_6, s_3) and (i_9, s_1) do not participate in a cycle, one seat of each of schools s_3 and s_1 remain to be inherited.

Step 3: The only student-seat pair remaining from Step 2 is (i_8, s_1). There is one seat of each of schools s_1, s_3, and s_4 to be inherited. Since there is still a student who is assigned one seat of school s_1 (namely student i_8 himself) at an earlier step, only one seat of school s_3 and one seat of school s_4 are inherited by student i_8. Thus, two new student-seat pairs now form: (i_8, s_3) and (i_8, s_4). The only cycle that forms at this step is $\{(i_8, s_3)\}$ (see Figure 3). Student i_8 is placed to school s_3. Two seats of school s_1 and one seat of school s_4 remain vacant. The outcome is the allocation marked with rectangles.

Just like TTCM, ETTCM too, is Pareto efficient.

PROPOSITION 11 *The equitable top trading cycles mechanism is Pareto efficient.*

Just like TTCM, ETTCM too, is strategy-proof.

PROPOSITION 12 *The equitable top trading cycles mechanism is strategy-proof.*

ETTCM not only shares the same nice properties with TTCM, it also has more to offer in terms of equity. To make our argument more concrete, we give examples to compare ETTCM with TTCM in terms of equity aspects. The advantage of ETTCM over TTCM is most apparent when we consider cycles consisting of two student-seat pairs.

Example 10 (a) ETTCM vs. TTCM when there are cycles consisting of two student-seat pairs: Let $I \equiv \{i_1, i_2, \ldots, i_7\}$ and $S \equiv \{s_1, s_2\}$

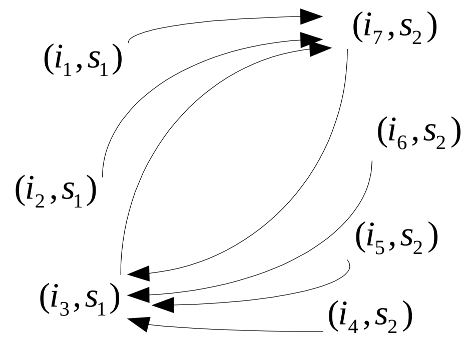

Figure 4: ETTCM applied to Example 10 (a)

where schools s_1 has three seats and schools s_2 has four seats. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$
i_1	i_4
i_2	i_5
i_3	i_6
i_7	i_7
i_5	i_3
i_6	i_1
i_4	i_2

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	P_{i_7}
$\underline{\boxed{s_2}}$	$\underline{\boxed{s_2}}$	$\underline{\boxed{s_2}}$	$\underline{s_1}$	$\underline{\boxed{s_1}}$	$\underline{\boxed{s_1}}$	$\boxed{s_1}$
s_1	s_1	s_1	$\boxed{s_2}$	s_2	s_2	$\underline{s_2}$

Here, students i_4, i_5, i_6, and i_7 have identical preferences and they are competing for one seat at school s_1. According to the priority order $\succ_{s_1}$, it is student i_7 who deserves one seat at school s_1 before any other student among the four students.

When we apply the TTCM algorithm to this problem, three cycles form in the first three steps. In these cycles, each of students i_1, i_2, and i_3 trades one seat of school s_1 for one seat of school s_2 with students i_4, i_5, and i_6 respectively. Student i_7 inherits the last seat of school s_2 and forms a self-cycle. The TTCM allocation is the underlined allocation above. Note that at this allocation, student i_7 has justified envy for all three students who have been placed to school s_1. Let us now apply the ETTCM algorithm to the same problem.

The first step of the ETTCM algorithm is depicted in Figure 4. The only cycle is $\{(i_3, s_1), (i_7, s_2)\}$. Student i_7 is placed to school s_1 and student i_3 to school s_2. Then these student-seat pairs are removed. In the second step, all student-seat pairs containing a seat of school s_1 point to the student-seat pair (i_5, s_2) because student i_5 has the highest priority for school s_1 among the remaining students who are assigned a seat of school s_2. All student-seat pairs containing a seat of school s_2 point to the student-seat pair (i_1, s_1) because student i_1 has the highest priority for school s_2 among the remaining students who are assigned a seat of school s_1. Then the only cycle is $\{(i_5, s_2), (i_1, s_1)\}$. Student i_5 is placed to school s_1 and student i_1 to school s_2. Continuing the algorithm in a similar way, we obtain the allocation marked with rectangles. Note that this allocation is fair. (Of course, this may not be the case in general.) Now student i_7 is better off whereas student i_4 is worse off as compared to the outcome of TTCM. By giving way more trading power to student i_4 than he deserves, TTCM severely violates the priorities of student i_7. However, ETTCM successfully establishes equity among students.

One can also observe that the size of justified envy caused by TTCM in the way described in Example 10 (a) increases as the number of seats available at schools increases. Next we argue that if the number of participants in a cycle is larger than two, the chances of causing justified envy among students is lower under ETTCM as compared to TTCM. To illustrate this point we give a simple example.

Example 10 (b) ETTCM vs. TTCM when there are cycles consisting of more than two student-seat pairs): Let $I \equiv \{i_1, i_2, i_3, i_4\}$ and $S \equiv \{s_1, s_2, s_3, s_4\}$ where schools s_1, s_3, and s_4 have only one seat and school s_2 has two seats. Students i_1 and i_2 have identical preferences. Thus, they are competing for the same seats.

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ'_{s_3}$
i_4	i_2	i_3	i_3
i_1	i_1	i_1	i_2
i_3	i_3	i_2	i_1
i_2	i_4	i_4	i_4

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
$\boxed{s_3}$	$\underline{s_3}$	$\boxed{s_1}$	$\boxed{s_2}$
s_1	s_1	s_2	s_1
$\underline{s_2}$	$\boxed{s_2}$	s_3	s_3

Let us first suppose that the priority order for school s_3 is $\succ_{s_3}$. The first steps of TTCM and ETTCM are depicted in Figure 5. Note that students

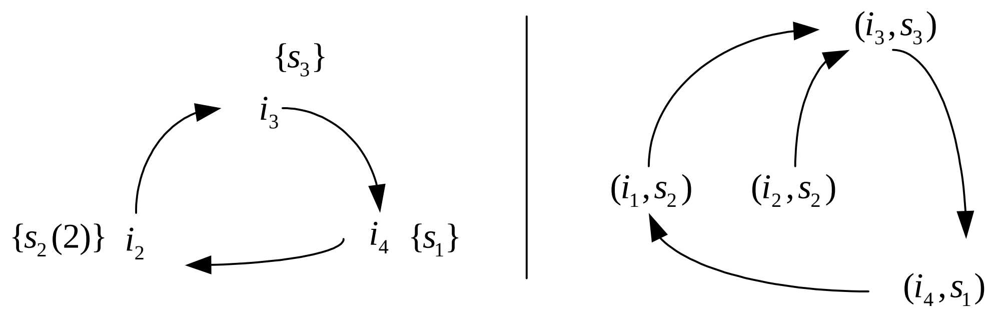

Figure 5: TTCM (left) vs. ETTCM (right)

i_1 and i_2 are competing for the only seat at school s_3. Clearly, TTCM places student i_2 to school s_3. The TTCM allocation is the underlined allocation above. At this allocation, student i_1 has justified envy for students i_2 and i_3 who are placed to schools s_3 and s_1 respectively. For the same problem note that the ETTCM allocation is the one marked with rectangles above. This allocation is fair.

Let us now suppose the priority order for school s_3 is $\succ'_{s_3}$. Since now student i_2 has higher priority for school s_3, at the TTCM allocation (which is the same as before) student i_1 has justified envy only for student i_3 who is placed to school s_1. On the other hand, at the ETTCM allocation (which is also the same as before) student i_2 has justified envy only for student i_1 who is placed to school s_3.

Recall that students i_1 and i_2 have identical preferences. In the two cases we considered, ETTCM does at least as well as TTCM. If the two priority orders for school s_3 are equally likely, then the expected number of justified envy situations are lower under ETTCM.

6 Conclusion

The equity criterion "reasonable fairness" we introduced here as a weaker requirement than the strong "fairness" criterion, enabled us to recover a critical compatibility between equity and optimality. Ergin and Sönmez (2003) argue that the deficiency of the widely-used Boston mechanism due to its lack of efficiency is even more serious than that of it due to strong vulnerability to strategic behavior. They argue that this large efficiency loss can be recovered by switching to the student optimal stable mechanism. The efficiency adjusted deferred acceptance mechanism we propose, however, eliminates the potentially large efficiency loss of the student optimal stable mechanism itself.

Although the efficiency adjusted deferred acceptance mechanism is not fully immune to strategic behavior, since students have very limited information about each other's preferences in almost all real-life applications, it recovers this property as well and in this sense, it is no weaker than the mentioned strategy-proof mechanisms. Achieving all three most desirable requirements that would be expected from an admissible mechanism, to quite a satisfactory degree, the efficiency adjusted deferred acceptance mechanism seems as a very promising practical tool for the placement of students to schools in real-life applications.

On the other hand, the main strengths of our second proposal, the equitable top trading cycles mechanism, is in terms of Pareto efficiency and strategy-proofness. Furthermore, by adopting a more picky procedure than its competitor, the top trading cycles mechanism (Abdulkadiroğlu and Sönmez, 2003), it attains superior equity aspects. It is also possible to modify both mechanisms to accommodate controlled choice issues (to be included in later versions of the paper).

7 Appendices

7.1 Appendix A

Example 1 (The detailed DA table):

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_1, i_2}, i_3$	$\boxed{i_4, i_5}, i_6$	$\boxed{i_7, i_8}, i_9$	$\boxed{i_{10}, i_{11}}, i_{12}$	
2	$\boxed{i_1, i_6}, i_2$	$\boxed{i_4, i_{12}}, i_5$	$\boxed{i_7, i_3}, i_8$	$\boxed{i_{10}, i_9}, i_{11}$	
3	$\boxed{i_{11}, i_6}, i_1$	$\boxed{i_8, i_{12}}, i_4$	$\boxed{i_2, i_3}, i_7$	$\boxed{i_5, i_9}, i_{10}$	
4	$\boxed{i_{11}, i_{10}}, i_6$	$\boxed{i_1, i_8}, i_{12}$	$\boxed{i_2, i_4}, i_3$	$\boxed{i_5, i_7}, i_9$	
5	$\boxed{i_9, i_{10}}, i_{11}$	$\boxed{i_1, i_3}, i_8$	$\boxed{i_4, i_{12}}, i_2$	$\boxed{i_6, i_7}, i_5$	
6	$\boxed{i_8, i_9}, i_{10}$	$\boxed{i_3, i_{11}}, i_1$	$\boxed{i_{12}, i_5}, i_4$	$\boxed{i_2, i_6}, i_7$	
7	$\boxed{i_4, i_8}, i_9$	$\boxed{i_{11}, i_7}, i_3$	$\boxed{i_{10}, i_5}, i_{12}$	$\boxed{i_1, i_2}, i_6$	
8	$\boxed{i_4, i_{12}}, i_8$	$\boxed{i_7, i_9}, i_{11}$	$\boxed{i_{10}, i_6}, i_5$	$\boxed{i_1, i_3}, i_2$	
9	$\boxed{i_{12}, i_5}, i_4$	$\boxed{i_2, i_9}, i_7$	$\boxed{i_6, i_{11}}, i_{10}$	$\boxed{i_8, i_3}, i_1$	
10	$\boxed{i_5, i_7}, i_{12}$	$\boxed{i_2, i_{10}}, i_9$	$\boxed{i_{11}, i_1}, i_6$	$\boxed{i_8, i_4}, i_3$	
11	$\boxed{i_5, i_7}$	$\boxed{i_2, i_{10}}$	$\boxed{i_{11}, i_1}$	$\boxed{i_8, i_4}$	$\boxed{i_3, i_6, i_9, i_{12}}$

7.1.1 Another reasonably fair and Pareto efficient mechanism

It is possible to obtain an alternative reasonably fair and Pareto efficient mechanism by combining the student optimal stable mechanism with the equitable top trading cycles mechanism. This can be achieved by the following two-stage algorithm applied to a given student placement problem:

Stage 1: Run the DA algorithm and obtain the allocation selected by the student optimal stable mechanism.

Stage 2: Assign each student one seat from the school he is placed to under the student optimal stable mechanism so that student-seat pairs form just like in the algorithm used to find the outcome of the equitable top trading cycles mechanism. Note that each student is contained in only one student-seat pair. Then apply the equitable top trading cycles algorithm.

One important point here is that unlike the case for the equitable top trading cycles algorithm, because now each student is part of only one student-seat pair, there is no inheritance of seats. One advantage of this new algorithm is that it is indeed an easy way of finding reasonably fair and Pareto efficient allocations.

PROPOSITION 13 *The mechanism that associates the outcome of the above algorithm to each student placement problem is reasonably fair and Pareto efficient.*

The reasonable fairness of this mechanism follows from the fact that at the ETTCM algorithm, no student-seat pair (i, s) ever points to a student-seat pair (i', s') where school s' is worse for student i than school s, and the fact that that school s is the placement of student i under SOSM. The Pareto efficiency of this mechanism follows from the fact that at the SOSM allocation, each student finds his placement at least as good as being placed to a school which still has unoccupied seats, and the fact that ETTCM is Pareto efficient. We omit the proofs.

7.2 Appendix B: Proofs of Section 4

PROOF OF LEMMA 1

Let φ be a Pareto efficient mechanism. Given a student placement problem $((\succ_s)_{s\in S}, (P_i)_{i\in I})$, let i be a student who prefers his placement under the student optimal stable mechanism to his placement under φ. Let s be the school student i is placed to under the student optimal stable mechanism and s' the school he is placed to under φ. Thus, $s \; P_i \; s'$. If student i does not have justified envy for another student who is placed to school s under φ, then all students who are placed to school s under φ have higher priority than student i for school s.[17] This means there is a student i' among them who is not placed to school s under the student optimal stable mechanism. Then student i' did not apply to school s at the DA algorithm because otherwise he would not be rejected. Then student i' also prefers his placement under the student optimal stable mechanism to his placement under φ.

Q.E.D.

PROOF OF PROPOSITION 7

LEMMA 2 *Given a student placement problem, the allocation obtained at*

[17] This is usually referred as the *individual rationality* property.

the end of a Round r, $r \geq 1$, of the algorithm to find the outcome of the efficiency adjusted deferred acceptance mechanism places each student to a school which is at least as good for him as the school he was placed to at the Round $r-1$.

PROOF OF LEMMA 2

Suppose by contradiction that there is a student placement problem, a Round r, $r \geq 1$, of the EADAM algorithm, and a student i_1 such that the school student i_1 is placed to at Round r is worse for him than the school s_1^{r-1} he was placed to at Round $r-1$. This means when we run the DA algorithm at Round r, student i_1 is rejected from school s_1^{r-1}. Then, there is a student $i_2 \in I \backslash \{i_1\}$ who is placed to school s_1^{r-1} at Round r and who was placed to a school s_2^{r-1} (at Round $r-1$) which is better for him than school s_1^{r-1}. Then, this means there is a student $i_3 \in I \backslash \{i_1, i_2\}$ who is placed to school s_2^{r-1} at Round r and who was placed to a school s_3 which is better for him than school s_2^{r-1} at Round $r-1$, and so on. Thus, there is a student $i_k \in I \backslash \{i_1, \ldots, i_{k-1}\}$ who is the *first* student to apply to a school s_{k-1}^{r-1} which is worse for him than the school s_k^{r-1} he was placed to at Round $r-1$.

We consider two cases:

Case 1. *Student i_k is not a blocker of Round $r-1$* : The preferences of student i_k is the same in Rounds r and $r-1$. Then, there is a student who is placed to school s_k^{r-1} at Round r and who did not apply to it at Round $r-1$. But then, this contradicts the assumption that student i_k is the first student to apply to a school which is worse for him than the school he was placed at Round $r-1$.

Case 2. *Student i_k is a blocker of Round $r-1$* : At Round r, student i_k, instead of applying to the school he is a blocker for, applied to his next choice, say school s^*. Student i_k had also applied to school s^* at Round $r-1$. Then, there is a student who is placed to school s_k^{r-1} at Round r and who did not apply to it at Round $r-1$. But then, this again contradicts the assumption that student i_k is the first student to apply to a school which is worse for him than the school he was placed to at Round $r-1$.

Q.E.D.

Hence, by Lemma 2, at each round of the algorithm, no student is placed to a school which is worse for him than the school he is placed to under the student optimal stable mechanism. Then the result follows from the definition of *reasonable fairness.*

Q.E.D.

PROOF OF PROPOSITION 8

Suppose by contradiction that there is a student placement problem for which the allocation selected by EADAM is not Pareto efficient. Let α denote this allocation. Hence, there is another allocation β that Pareto dominates allocation α. Suppose the algorithm terminates in R rounds. Given $r \in \{1, 2, \ldots, R\}$, let α_r denote the allocation obtained at the end of Round r of the algorithm. By Lemma 2, allocation β also Pareto dominates each allocation α_r, $r \in \{1, 2, \ldots, R\}$.

We first show that at allocation β, no blocker of a Round r, $r \in \{1, 2, \ldots, R\}$, is placed to the school for which he is a blocker at Round r. We argue by induction. Suppose, at allocation β, there is a blocker i_1 of Round 1 who is placed to a school s_1 for which he is a blocker at this round. Note that at allocation α_1, all the seats of school s_1 are full. (Otherwise, student i_1 would not be rejected from it at Round 0.) Since allocation β Pareto dominates allocation α_1, there is a student i_2 who is placed to school s_1 at allocation α_1, and who is placed to a school s_2 which is better for him, at allocation β. Note again that at allocation α_1, all the seats of school s_2 are also full. Then, there is a student i_3 who is placed to school s_2 at allocation α_1, and who is placed to a school s_3 which is better for him, at allocation β. Continuing in a similar way, we conclude that because allocation β Pareto dominates allocation α_1, there is a student i_k who is placed to school s_{k-1} at allocation α_1, and who is placed to school s_1 which is better for him, at allocation β. That is, there is a cycle of students $(i_1, i_2, \ldots, i_k)$ such that each prefers the school the next student in the cycle (for student i_k it is i_1) is placed to at allocation α_1, to the the school he is placed to at the same allocation. Let us consider the DA algorithm ran in Round 0 of the algorithm. Let $i \in \{i_1, i_2, \ldots, i_k\}$ be the student in this cycle who is the *last* (or, one of the last, if there are more than one such students) to apply to the school, say school s, he is placed to at the end of this round. Then the student in the above cycle who prefers school s to the school he is placed to at allocation α_1, was rejected from there at an earlier step. Then, when student i applies to school s, all the seats are already full and since student i is placed to this school at the end of the round, some student i' is rejected. Then, student i' is a blocker for school s. Furthermore, by the assumption about student i, student i' is rejected from school s, at a step later than the step the blocker i_1 is rejected from school s_1 for which he is a blocker. But then, student i_1 can not be a blocker of

Round 1.

Suppose that at allocation β, no blocker of a Round k, $1 \leq k \leq r-1$, is placed to the school for which he is a blocker at Round k. We want to show that at allocation β, no blocker of Round r is placed to the school for which he is a blocker at Round r. Consider allocation α_r. Since β Pareto dominates each allocation α_r, using the same argument as in the previous paragraph, there is a cycle of students $(i'_1, i'_2, \ldots, i'_k)$ such that each prefers the school the next student in the cycle (for student i'_k it is i'_1) is placed to at allocation α_r, to the the school he is placed to at the same allocation. Furthermore, due to our supposition about the blockers of earlier rounds, none of the students in this cycle is a blocker for the school he prefers. Then, for each of the students in the cycle there is a corresponding step of the DA algorithm ran in Round $r-1$ such that he is rejected from the school he prefers, at that step. But then, we can again apply the same argument we used in the previous paragraph to conclude that student i'_1 can not be a blocker of Round r.

At the end of Round R of the algorithm, there are no blockers left and we obtain the allocation α. Since allocation β Pareto dominates allocation α, there is again a cycle $(i''_1, i''_2, \ldots, i''_k)$ of students who prefers the school the next student in the cycle (for student i''_k it is i''_1) is placed to at allocation α to the the school he is placed to at the same allocation. Note that by what we just proved in the previous paragraph, no student in $\{i''_1, i''_2, \ldots, i''_k\}$ can be a blocker at any round for the school the next student in the cycle is placed to at allocation α. Therefore, each student in $\{i''_1, i''_2, \ldots, i''_k\}$ applies to the the school the next student in the cycle is placed to at a step of the DA algorithm ran in Round R. Let $i'' \in \{i''_1, i''_2, \ldots, i''_k\}$ be the student in this cycle who is the *last* to be rejected from the school the next student in the cycle is placed to. When student i'' applies to the school he is placed to at the end of Round R, because the student in $\{i''_1, i''_2, \ldots, i''_k\}$ who prefers this school to the school he is placed to was rejected from here at an earlier step, there is a student i''' who is rejected from this school. Then student i''' is a blocker for this school, contradicting this round being the last round.

Q.E.D.

PROOF OF PROPOSITION 9

Consider the student placement problem given in Example 8. When each student truthfully submits his preferences, then the outcome of the DA algorithm is the underlined allocation which is Pareto efficient. Hence, any reasonably fair and Pareto efficient mechanism has to select this allocation. Suppose student i_2 submits fake preferences P'_{i_2}. Then the outcome of the DA algorithm for the new problem is the twice underlined allocation which is not Pareto efficient. But the unique Pareto efficient allocation that Pareto dominates this allocation places student i_2 to school s_1. Thus, student i_2 gains by submitting the fake preferences.

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_3	i_2	i_2
i_1	i_3	i_1
i_2		

P_{i_1}	P_{i_2}	P'_{i_2}	P_{i_3}
$\underline{s_1}$	s_1	s_1	$\underline{s_3}$
$\underline{\underline{s_2}}$	$\underline{s_2}$	$\underline{\underline{s_3}}$	$\underline{\underline{s_1}}$
	s_3	s_2	

Q.E.D.

PROOF OF PROPOSITION 10

Before, we prove Proposition 10, following Ehlers (2002b), we formalize the discussion in the text. First, we note that although Ehlers (2002b) does his analysis for the case when each hospital is hiring one worker, his result generalizes to our case, too. Second, we assume that all schools are acceptable, which also does not affect the result, either. In this strategic analysis, each student is now a player:

Given $s \in S$, let $\mathcal{B}_s$ be the class of all strict priority orders for school s. Given $i \in I$, let $\mathcal{P}_i$ be the class of all strict preferences for student i. Let $\mathcal{X}_{-i} \equiv (\mathcal{B}_s)_{s \in S} \times (\mathcal{P}_{i'})_{i' \in I \setminus \{i\}}$. A random preference profile is a probability distribution $\widetilde{P}_{-i}$ over $\mathcal{X}_{-i}$. Here, $\widetilde{P}_{-i}$ is interpreted as student i's belief (or, his information) about the stated preferences of the other students. Let $\mathcal{A}$ be the set of all allocations. A random allocation $\widetilde{a}$ is a probability distribution over $\mathcal{A}$. Let $\widetilde{a}(i)$ be the distribution which $\widetilde{a}$ induces on the set of student i's placements S. Let φ be a mechanism. Given a student placement problem (P_i, P_{-i}) where $P_{-i} \in \mathcal{X}_{-i}$, let $\varphi(P_i, P_{-i})$ be the allocation selected by φ for this problem. Also, let $\varphi(P_i, P_{-i})(i)$ denote student i's placement at this allocation. Given a mechanism φ and a student i with preferences P_i, each random preference profile $\widetilde{P}_{-i}$ induces a random allocation $\varphi(P_i, \widetilde{P}_{-i})$ in the following way: for all $a \in \mathcal{A}$, $\Pr\{\varphi(P_i, \widetilde{P}_{-i}) = a\} = \Pr\{\widetilde{P}_{-i} = P_{-i} \text{ and } \varphi(P_i, P_{-i}) = a\}$. Let $\varphi(P_i, \widetilde{P}_{-i})(i)$ be the distribution which $\varphi(P_i, \widetilde{P}_{-i})$ induces over student i's set of placements. Given $i \in I$, P_i, P'_i, $P''_i \in \mathcal{P}_i$, and a random preference profile $\widetilde{P}_{-i}$, we say that strategy P'_i **stochastically** P_i-**dominates** strategy P''_i if for all $s \in S$, $\Pr\{\varphi(P'_i, \widetilde{P}_{-i})(i)\ R_i\ s\} \geq \Pr\{\varphi(P''_i, \widetilde{P}_{-i})(i)\ R_i\ s\}$.

We consider a model where a student can not distinguish between two schools (*i.e.*, is not sure about how other students rank the two schools). In such a case, we say that his information about the two schools is symmetric. Then , such a student believes that any student placement problem is equally likely as its symmetric problem in which the roles of the two schools are exchanged. Formally (Roth and Rothblum, 1999), given $i \in I$, $P_i \in \mathcal{P}_i$, and s, $s' \in S$, let $P_i^{s \leftrightarrow s'}$ denote the preferences in which the positions of s and s' are exchanged and the other positions in P_i are unchanged. Let $P_{-i}^{s \leftrightarrow s'}$ denote the profile such that each student $i' \in N \backslash \{i\}$ exchanges the positions of s and s' in his preferences, schools s and s' exchange their priority orders (*i.e.*, $\succ_s$ becomes the priority order for school s' in $P_{-i}^{s \leftrightarrow s'}$, and $\succ_{s'}$ becomes the priority order for school s in $P_{-i}^{s \leftrightarrow s'}$), and the priority orders of the other schools remain unchanged. Given $i \in I$ and s, $s' \in S$, student i's information for schools s and s' is **symmetric** if P_{-i} and $P_{-i}^{s \leftrightarrow s'}$ are equally probable, *i.e.*, $\Pr\{\widetilde{P}_{-i} = P_{-i}\} = \Pr\{\widetilde{P}_{-i} = P_{-i}^{s \leftrightarrow s'}\}$.

Next, we define the two conditions (given in Theorem 1 of Ehlers, 2002b) that would establish the result in Proposition 10. Given $a \in \mathcal{A}$ and s, $s' \in S$, let $a^{s \leftrightarrow s'}$ denote the allocation such that for all $i \in I$, (i) if $a(i) \notin \{s, s'\}$, then $a^{s \leftrightarrow s'}(i) = a(i)$, (ii) if $a(i) = s$, then $a^{s \leftrightarrow s'}(i) = s'$, and (iii) if $a(i) = s'$, then $a^{s \leftrightarrow s'}(i) = s$.

Neutrality: For all $i \in I$, all $P_i \in \mathcal{P}_i$, all $P_{-i} \in \mathcal{X}_{-i}$, and all s, $s' \in S$, if $\varphi(P_i, P_{-i}) = a$, then $\varphi(P_i^{s \leftrightarrow s'}, P_{-i}^{s \leftrightarrow s'}) = a^{s \leftrightarrow s'}$.

It is easy to see that EADAM satisfies neutrality. Next we define the second requirement of Ehlers (2002b). It says that if the position of the school s a student is placed to is exchanged with that of another school s' which he prefers to school s, then the student's placement should not change.

Positive Association: For all $i \in I$, all $P_i \in \mathcal{P}_i$, all $P_{-i} \in \mathcal{X}_{-i}$, and all s, $s' \in S$, if $\varphi(P_i, P_{-i})(i) = s$ and $s' \; P_i \; s$, then $\varphi(P_i^{s \leftrightarrow s'}, P_{-i})(i) = s$.

First of all, note that SOSM already satisfies this requirement. This is because when the ranking of the school he is placed to improves in the preferences of a student i, because other students' preferences (as well as the priority orders) remain unchanged, at the new problem (*i.e.*, at the problem where the two positions of the two schools in student i's preferences are exchanged) no student applies to a school he did not apply to at the initial student placement problem (*i.e.*, at the problem where the two positions of the two schools in student i's preferences were not exchanged) and moreover, a student may even apply to less schools now. Also, note that the DA algorithms for the two problems are identical until the step of the DA algorithm (applied to the new problem) at which student i applies to the school he is placed to at the initial problem. Then student i can not be rejected from that school and he is placed to the same school at the new problem. An important observation here is that the school a student is placed to is not affected by which schools he was rejected from before applying to that school.[18]

The fact that EADAM satisfies positive association can intuitively be seen as follows. EADAM algorithm mimics the DA algorithm to a certain extent. Suppose a student i is placed to a school s at the EADAM outcome. By Proposition 7, at Round 0, he must have applied to school s at some step of the DA algorithm and he could have been rejected from school s at a later step in that round. If student i was rejected from school s at some step of the DA algorithm ran in Round 0, then by the way the EADAM algorithm is designed, he is placed back at school s at a later round, because of the schools he applied to after school s. That is student i's choices after school s matter for determining his placement at the EADAM outcome (not those he applied to before applying to that school!). When we exchange the positions of schools s and s' where $s' \; P_i \; s$ in student i's preferences, the ranking of the schools which he used to value worse than school s are unaffected. We consider two cases:

Case 1. *At the end of Round 0, student i is placed to school s* : Then the same idea why SOSM satisfies positive association applies and student i's placement does not change.

Case 2. *At the end of Round 0, student i is placed to a worse school for him than school s* : At Round 0 of EADAM algorithm applied to the new problem, student i is again rejected from school s. After being rejected from school s, because other students' preferences are unaffected, student i is also rejected from the schools he used to rank between schools s and s' (and he can not initiate any rejection chains that would cause any student to be identified as a blocker for these schools since if this was possible, it would happen at the initial problem). Since the ranking of the schools which student i used to value worse than school s are unaffected, the outcome of EADAM does not change.

[18] This may affect the placement of other students, though.

Q.E.D.

7.3 Appendix C: Proofs of Section 5

LEMMA 3 *Given a student placement problem and a step of the ETTCM algorithm, if the best choice of a student i among the remaining seats is a seat at school s, then student i does not ever point to a pair which contains a seat from a different school until no vacant seats at school s remain.*

PROOF OF LEMMA 3

This is due to the fact that for each school s whose seats remained to be inherited from earlier steps, the to-be-inherited seats are assigned to the remaining students, right after the step at the end of which no student who was assigned a seat of school s at an earlier step is left.

Q.E.D.

PROOF OF PROPOSITION 11

A critical observation about the ETTCM algorithm is that by Lemma 3, if a school still has vacant seats at a step of the ETTCM algorithm, then there is a student-seat pair containing a seat from that school at that step. Then, the idea behind the Pareto efficiency of ETTCM is the same as that of TTCM. Given a student placement problem, each student who leaves at the first step is placed to his best choice, hence he can not be made better off. Each student who leaves at the second step is placed to his best choice among the remaining seats, hence he can not be made better off without making someone who left at the first step worse off. Continuing in this way, no student can be made better off without making someone who left at an earlier step worse off.

Q.E.D.

LEMMA 4 *Given a student placement problem $((\succ_s)_{s\in S}, (P_i)_{i\in I})$, suppose a student i is removed at a Step t of the ETTCM algorithm and, if he submits preferences R'_i instead of R_i, then he is removed at a Step t'. Then the remaining students, available seats of schools, and to-be-inherited seats of schools at the beginning of Step* $\min\{t,t'\}$ *are the same.*

PROOF OF LEMMA 4

Given a student placement problem $((\succ_s)_{s\in S}, (P_i)_{i\in I})$, because no student-seat pair containing student i participates in a cycle before Step $\min\{t,t'\}$, the same cycles form until Step $\min\{t, t'\}$ and the same students are placed to the same seats until Step $\min\{t, t'\}$.

Q.E.D.

LEMMA 5 *Given a student placement problem, if a student-seat pair* (i, s) *is pointing to another student-seat pair* (i', s') *at some step of the ETTCM algorithm, then student-seat pair* (i, s) *keeps pointing to the student-seat pair* (i', s') *as long as student* i' *is not removed.*

PROOF OF LEMMA 5

Given a student placement problem, if a student-seat pair (i, s) is pointing to another student-seat pair (i', s') at a Step t, then school s' is the best choice of student i among the remaining schools. Furthermore, student i' is the student among those who are assigned a seat from school s' who has the highest priority for school s. By Lemma 3, student i will keep pointing to a pair containing a seat from school s' until no vacant seats at school s remain. Then the only case student-seat pair (i, s) can point to another student-seat pair (i'', s'') before student i' is removed is when (i) $s'' = s'$ and (ii) student i'' has higher priority than student i' for school s. But this is only possible if student i'' is assigned a seat of school s through inheritance at some Step t', $t' > t$. But no inheritance of seats of school s' takes place before student i is removed.

Q.E.D.

PROOF OF PROPOSITION 12

Given a student placement problem $((\succ_s)_{s\in S}, (P_i)_{i\in I})$ and a student i, let t be the step at which student i is removed and s be the school he is placed to. We will show that if student i submits fake preferences R'_i, he can not be placed to a school which is better for him than school s. Let t' be the step at which student i is removed when he submits R'_i and s'' be the school he is placed to at this step. We consider two cases:

Case 1. $t \geq t'$: Consider Step t'. By Lemma 4, at the beginning of this step, the remaining students and seats are the same. Note that by Lemma 5, each student-seat pair (i', s') that is pointing to a pair containing student i keeps pointing to that pair as long as student i stays. Similarly, each student-seat pair (i'', s'') that is pointing to the pair (i', s') keeps pointing to that pair as long as student i' stays which is the case as long as student i stays, and so on. Consequently, at Step t, student i has the opportunity to participate in any of the cycles he participates under fake preferences R'_i. Since under his true preferences, he is pointing to his best choice at Step t, school s can not be worse than school s' for student i.

Case 2. $t < t'$: By Lemma 4, at the beginning of Step t, the remaining students and seats are the same. Since student i is placed to his best choice at this step, he can not be placed to a better school at a later step.

Q.E.D.

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