Individual Powers and Social Consent: an Axiomatic Approach

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March 28, 2006

Abstract

We introduce a notion of conditionally decisive powers of which the exercise depends on social consent. Decisive powers, or the so-called libertarian rights, are examples and much weaker forms of powers are covered by our notion. We provide an axiomatic characterization of existence of a system of powers and its uniqueness as well as characterizations of various families of rules represented by systems of powers. Critical axioms are *monotonicity*, *independence*, and *symmetric linkage* (person i and i's issues should be treated symmetrically to person j and j's issues for at least one linkage between issues and persons). Considering a domain of simple preference relations (trichotomous or dichotomous preferences), we show under a certain assumption on the model that a rule satisfies *Pareto efficiency*, *independence*, and *symmetric linkage* if and only if it is represented by a "quasi-plurality system of powers". For the exercise of a power under a quasi-plurality system, at least either a majority (or $(n+1)/2$) consent or a 50% (or $(n-1)/2$) consent is needed.

Keywords: Powers; Consent; Libertarian Rights; Monotonicity; Independence; Symmetric linkage; Pareto efficiency; Plurality

JEL Classification Numbers: D70, D71, D72

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1 Introduction

Numerous decision rules in social or political institutions feature some sorts of individual or positional powers. Exercising these powers is often conditional upon obtaining sufficient social consent and the level of the sufficiency may vary across powers. To take an example, the Constitution of United States describes powers of the President and how much degree of social consent is required for exercising presidential powers; for instance, '*power*, by and with the advice and *consent* of the Senate, to make treaties, provided two thirds of the Senators present concur'.[1] The main objective of this paper is to formalize a notion of individual powers of which the exercise depends on social consent and to give axiomatic characterizations of some families of rules represented by a system of powers.[2]

We consider a simple opinion aggregation model. There is a society consisting of at least two members. There are a finite number of issues. The society needs to make a decision on each issue either positively (acceptance) or negatively (rejection), reflecting members' opinions that are expressed in one of the three ways, positively or negatively or neutrally (we also consider separately the case when opinions are either positive or negative). A decision *rule* associates with each profile of members' opinions, namely, a *problem*, a profile of decisions on the issues. An example is the model of qualification problems studied by Kasher and Rubinstein (1997) and Samet and Schmeidler (2003), where a society needs to identify a group of qualified members.

Building on Samet and Schmeidler (2003), we say that person i has the power on the k^{th} issue if social decision on the k^{th} issue is made according to i's opinion when and only when i's opinion obtains sufficient social consent. The sufficiency means that the number of persons with the same opinion on the k^{th} issue as i's is greater than or equal to a certain level, called a *consent quota*.[3] For example, decisive powers, or the so-called libertarian rights by Sen (1970, 1976) and Gibbard (1974), have the minimum consent quota of 1. The above mentioned Presidential power has the consent quota of 2/3 of the number of the Senators. A *system of powers* is a function associating with each issue a person who has the power on this issue and the corresponding consent quotas.

Samet and Schmeidler (2003, Theorem 1), in our terminology, provide an axiomatic characterization of existence of a system of powers which gives each person the power of qualifying himself. We extend this result in our generalized model by considering the following modification of their three main axioms. *Monotonicity* says that the rule should respond non-negatively whenever the set of members with the positive opinion on each issue expands and the set of members with the negative opinion shrinks. *Independence* says that the decision on each issue should be based only on members' opinions on this issue and not on their opinions on the other issues.[4] A rather drastic modification is in their symmetry axiom. We consider an environment where issues have some connections or linkages with

[1] The United States Constitution, Article II, Section 2, Clause 2.

[2] Thomson (2001) offers an extensive survey and useful guidelines for the axiomatic method in game theory.

[3] When i's opinion is neutral, this description does not match exactly to our definition because decision on each issue cannot be neutral.

[4] These two axioms are also studied by Rubinstein and Fishburn (1986), Kasher and Rubinstein (1997), and Ju (2003, 2005).

persons. For example, each person has his own areas of specialty and each issue falls on an area of at least one person. Our model has a given (non-empty) set of possible linkages associating with each issue a person. *Symmetric linkage* says that the rule should treat a person i and i's areas symmetrically to any other person j and j's areas, under at least one *linkage* in the model. Samet and Schmeidler's (2003) model of qualification problems has the *unique* linkage associating each person with the issue of qualifying the person himself. For this reason, *symmetric linkage* in this special model reduces to their symmetry axiom.

We show that a rule satisfies *monotonicity*, *independence*, and *symmetric linkage* if and only if there is a system of powers representing the rule and that the system is unique up to a natural equivalence relation. Adding *anonymity* (names of opinion holders should not matter), we establish a necessary and sufficient condition for existence of a *non-exclusive system of powers*, under which everyone has the equal power on every issue. Adding *neutrality* (names of issues should not matter either) instead of *anonymity*, we characterize rules represented either by a constant non-exclusive system of powers ("constant" means constant consent quotas across issues) or by a *monocentric system of powers*, under which one and only one person has powers on all issues. Finally, considering simple preference relations called dichotomous and trichotomous preference relations (Ju 2003, 2005), we show that rules represented by "quasi-plurality systems of powers" are the only rules satisfying *Pareto efficiency*, *independence*, and *symmetric linkage*. This result is established under a certain assumption on the set of possible linkages in the model, which holds in Samet and Schmeidler's (2003) model among others. Dropping this assumption but adding *neutrality*, we establish a similar characterization result. Under a *quasi-plurality system*, exercising a power needs at least either majority (or $(n+1)/2$) consent or 50% (or $(n-1)/2$) consent.

Our definition of a system of powers allows for a wide spectrum of examples that were not captured in the earlier studies. On the one extreme, we have monocentric systems of powers giving only a single person powers on all issues. On the other extreme, we have non-exclusive systems of powers giving everyone the equal power on every issue. We also find that on the trichotomous domain, rules represented by a system of powers may quite differ from plurality rule, while, on the dichotomous domain, they are close to plurality (or majority) rule. Much richer variety of rules emerge after admitting neutral opinions. Incorporating neutral opinions, we think, is important because neutral opinions are common in realistic decision procedures (abstention can be viewed as an expression of a neutral opinion).

When issues are associated with personal matters such as believing in a religion, planting a tree in one's own backyard, etc., our powers and systems of powers can be interpreted as a weak notion of rights and systems of rights. In the Arrovian framework, Sen (1970, 1976, 1983) and many of his critics formulate individual rights based on (i) existence of the so-called recognized personal spheres (Gaertner, Pattanaik, and Suzumura 1992), and (ii) individuals' *decisiveness* on personal spheres (social decision on an issue in someone's sphere is decided by the person himself). Our definition of a system of powers is similar to this formulation with regard to aspect (i). This is because a system of powers links issues with persons who have the powers on these issues. However, with regard to aspect (ii), our definition is substantially weaker and flexible. Our powers, interpreted as rights, are just rights to *influence* social decision, not necessarily decisive but conditionally decisive (decisiveness is one extreme case in our definition). They are alienable as in Blau (1975) and Gibbard (1974).

But, alienation of rights in this paper relies on the degree of social consent.

Motivation for our weakening decisiveness component in the earlier definition comes, first of all, from realistic rights that are often conditionally decisive. For example, consider rights for smoking or for clean air. There are some places where smoking is prohibited and other places where smoking is allowed. A person's desire is not decisive in his own smoking. In order for a person to exercise his right, he needs to find a place where his desire can get sufficient consent from others. Motivation comes also from the so-called paradox of Paretian liberal. As pointed out by Sen (1970, 1976, 1983), Gibbard (1974) and other subsequent works,[5] existence of decisive rights is incompatible with *Pareto efficiency*. Sen (1983, p.14) proposed studying this compatibility issue in restricted preferences domains. However, we show that the paradox prevails even on the extremely restricted domains of trichotomous preferences (or dichotomous preferences). Thus, unless we are willing to abandon *Pareto efficiency*, it is inevitable to think about weakening "decisiveness" component in the definition of rights. How much weakening is necessary to escape from the paradox? Our characterization of the quasi-plurality systems shows that the weakening should be substantial because a person's power only has a tie-breaking role when the number of persons supporting an issue equals the number of persons opposing it; when there is no such tie, the social decision is ruled by plurality.

The rest of the paper is organized as follows. In Section 2, we define the model and basic concepts. In Section 3, we define main axioms. In Section 4, we state preliminary results. In Section 5, we state main results. Some proofs are relegated to the appendix for smooth passage.

2 Model and Basic Concepts

Let $N \equiv \{1, \cdots, n\}$, $n \geq 2$, be the set of *persons* and $M \equiv \{1, \cdots, m\}$ the set of *issues*. Each person $i \in N$ has his *opinion* on issues in M, represented by an $1 \times m$ row vector P_i consisting of 1, 0, or -1.[6] A *problem* is an $n \times m$ opinion matrix P consisting of n row vectors $P_1, \cdots, P_n$. Let $\mathcal{P}_{\text{Tri}}$ be the set of problems, called, the *trichotomous* (opinion) *domain*. An *alternative* is a list of either positive or negative decisions on all issues, formally, a vector of 1 and -1, $x \equiv (x_1, \cdots, x_k) \in \{-1, 1\}^M$, where 1 (resp. -1) in the k^{th} component means accepting the k^{th} issue (resp. rejecting the k^{th} issue). For each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$, P^k denotes the k^{th} column vector of P. Let

$$||P^k_+|| \equiv \sum_{i \in N: P_{ik}=1} P_{ik} \ , \ ||P^k_-|| \equiv \sum_{i \in N: P_{ik}=-1} -P_{ik} \ , \text{ and } ||P^k_{+,-}|| \equiv ||P^k_+|| + ||P^k_-||.$$

Let $\mathcal{P}_{\text{Di}}$ be the subset of $\mathcal{P}_{\text{Tri}}$, consisting of the opinion matrices whose entries are either 1 or -1, called the *dichotomous* (opinion) *domain*. Let $\mathcal{D}$ be either one of the two domains.

[5]See Deb, Pattanaik, and Razzolini (1997) for the paradox in a framework where rights are represented as a game form.

[6]Notation 'P' for 'o**P**inion'.

Samet and Schmeidler (2003) consider the dichotomous domain in qualification problems.[7]

A *decision rule* on $\mathcal{D}$, $f\colon \mathcal{D} \to \{-1,1\}^M$, associates with each problem in the domain a single alternative. We are interested in rules that are represented by a "system of powers" defined as follows. We present the definition, first, focusing on dichotomous opinions. After this, we give the general definition.

Given a rule f defined on the dichotomous domain $\mathcal{P}_{\text{Di}}$, person $i \in N$ has the "power to influence the social decision on the k^{th} issue", briefly, the *power on the k^{th} issue* if the decision on the k^{th} issue is made following person i's opinion whenever person i's opinion obtains sufficient consent from society: formally, there exist $q_+, q_- \in \{1, \cdots, n+1\}$ such that for each $P \in \mathcal{P}_{\text{Di}}$,

$$\begin{array}{l}\text{(i) when } P_{ik} = 1,\ \ f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_+ \ ; \\ \text{(ii) when } P_{ik} = -1,\ \ f_k(P) = -1 \Leftrightarrow ||P_-^k|| \geq q_- \ .\end{array} \tag{1}$$

The two numbers q_+ and q_- are called *consent-quotas.* The greater q_+ or q_- is, the higher social consent is required for the exercise of the power. There are two extreme cases. When $q_+ = q_- = 1$, i's opinion determines social decision independently of social consent. Thus the power is *decisive.* When $q_+ = n+1$ and $q_- = n+1$, the power is *void* because i's opinion is never reflected in the social decision.

The total number of positive or negative votes always equals n on the dichotomous domain. However, on the trichotomous domain, it is variable. Thus, we allow consent-quotas to vary relative to the total number of votes. Given a rule f defined on $\mathcal{P}_{\text{Tri}}$, a person $i \in N$ has the *power on the k^{th} issue* if there exist three functions $q_+\colon \{1,\ldots,n\} \to \{1,\ldots,n+1\}$, $q_0\colon \{0,1,\ldots,n-1\} \to \{0,1,\ldots,n\}$, and $q_-\colon \{1,\ldots,n\} \to \{1,\ldots,n+1\}$ such that for each $\nu \in \{0,1,\ldots,n\}$, $q_+(\nu)$, $q_0(\nu)$ and $q_-(\nu)$ are in $\{0,1,\ldots,\nu+1\}$, and for each $P \in \mathcal{P}_{\text{Tri}}$ with $||P_{+,-}^k|| = \nu$ (thus, ν denotes the number of positive or negative votes),

$$\begin{array}{l}\text{(i) when } P_{ik} = 1,\ \ f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_+(\nu) \ ; \\ \text{(ii) when } P_{ik} = 0,\ \ f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_0(\nu) \ ; \\ \text{(iii) when } P_{ik} = -1,\ \ f_k(P) = -1 \Leftrightarrow ||P_-^k|| \geq q_-(\nu) \ .\end{array} \tag{2}$$

We call $q(\cdot) \equiv (q_+(\cdot), q_0(\cdot), q_-(\cdot))$ the *consent-quotas function* (with a slight abuse of notation).[8] Let $\mathbf{Q}$ be the family of consent-quota functions. The power is *decisive* if for each ν, both $q_+(\nu)$ and $q_-(\nu)$ take the value of 1. The power is *void* if for each ν, both $q_+(\nu)$ and $q_-(\nu)$ take the value of $\nu+1$.

Definition 1 (System of Powers)**.** A *system of powers* representing a rule f on $\mathcal{P}_{\text{Tri}}$ is a function $W\colon M \to N \times Q$ mapping each issue $k \in M$ a pair of the person, $W_1(k)$, who has the power on the k^{th} issue, and the consent-quotas function, $W_2(k) = (q_+(\cdot), q_0(\cdot), q_-(\cdot))$, associated with the power.[9] That is, when $W_1(k) = i$, for each $\nu \in \{0,1,\ldots,n\}$ and each

[7]Dichotomous opinions in Samet Schmeidler (2003) are described by vectors of 1 and 0, where number 0 has the same meaning as -1 in our model.

[8]Since the three component functions q_+, q_0, q_- have different domains, q cannot be described as a function. But, including 0 in the domain of q_+ and q_- and defining the values at 0 arbitrarily will not make any difference and, this way, the problem can be avoided.

[9]Notation 'W' for 'po**W**er'.

	Issue 1	Issue 2
John	1	-1
Paul	1	1
Others	1	-1
Decision	1	-1

(a)

	Issue 1	Issue 2
John	1	1
Paul	-1	1
Others	-1	1
Decision	1	1

(b)

Table 1: When issue 1 is in John's area and issue 2 is in Paul's, the social decisions in the two cases exhibit a violation of *symmetric linkage*.

$P \in \mathcal{P}_{\text{Tri}}$ with $||P^k_{+,-}|| = \nu$, the social decision on the k^{th} issue is made as described in (2).

A rule may be represented by multiple systems of powers, although all these systems will be shown to be equivalent under a natural equivalence relation to be defined in Section 4.

3 Axioms

In this section, we define axioms for rules, which are crucial in this paper.

The first axiom says that rules should not respond negatively when the opinion matrix increases.

Monotonicity. For each $P, P' \in \mathcal{D}$, if $P \leqq P'$, $f(P) \leqq f(P')$.

The second axiom says that decisions on different issues should be made independently: decision on the k^{th} issue should rely only on the k^{th} column of the opinion matrix.

Independence. For each $P, P' \in \mathcal{D}$ and each $k \in M$, if $P^k = P'^k$, $f_k(P) = f_k(P')$.

We refer readers to Rubinstein and Fishburn (1986), Kasher and Rubinstein (1997), and Samet and Schmeidler (2003) for more discussion on the two axioms.

To introduce the next axiom, suppose that members of society have their own areas of specialty and each issue lies in some of these areas. Ideally, it is important that society treats all members and their areas of specialty in a symmetric manner. To illustrate this idea, suppose that the first issue is in John's area and the second issue is in Paul's. Consider the case depicted in Table 1-(a). Both John and Paul have positive opinions on their own issues and John is negative on Paul's issue while Paul is positive on John's issue. Everyone else is positive on John's issue and negative on Paul's. Suppose, as in the bottom row of Table 1-(a), that the social decision on Paul's issue, in this case, is against Paul's opinion (so negative). Now consider the case when John and Paul face the reverse situation as depicted in Table 1-(b), that is, John faces the same situation regarding his area as Paul faced in the earlier case. If the social decision on John's issue in this case follows John's opinion (so it differs from the decision on Paul's issue in the earlier case), one could argue that the rule favors John and John's area relative to Paul and Paul's area. Our next axiom prevents such an asymmetric treatment. An issue may lie in multiple areas and so there may exist multiple linkages between issues and persons. Requiring symmetric treatment with respect to all possible linkages can be too strong. The next axiom requires symmetric treatment for at least one linkage.

To define this axiom formally, call a function $\lambda\colon M \to N$ mapping each issue into a person a *linkage*. Let $\boldsymbol{\Lambda}$ be the non-empty *set of possible linkages* in the model. A linkage may have different interpretations depending on applications. In the above mentioned application, a linkage describes areas of specialties. In the qualification problem studied by Samet and Schmeidler (2003), $M = N$ and the identity function from N to N is the linkage describing the nominal correspondence between person i and the qualification of i.[10] When issues in M are proposals made by some members in N, a linkage may describe who proposed what issues. When the problem is to approve candidates in M, a linkage may describe a personal relation between candidates in M and voters in N. When issues are private properties, a linkage may describe initial ownership (who owns what properties).

The next axiom says that for at least one linkage in Λ, the rule should treat each person i and i's issues symmetrically to any other person j and j's issues. Technically, when names of person i and all i's issues are switched simultaneously to names of person j and all j's issues, social decision should also be switched accordingly. Given a linkage $\lambda \in \Lambda$, for each $i \in N$, let us call elements in $\lambda^{-1}(i)$ person i's issues. Let $\pi\colon N \to N$ and $\delta\colon M \to M$ are permutations on N and on M such that δ maps the set of each person i's issues *onto* the set of person $\pi(i)$'s issues. Let ${}^{\delta}_{\pi}P$ be the matrix such that for each $i \in N$ and each $k \in M$, ${}^{\delta}_{\pi}P_{ik} \equiv P_{\pi(i)\delta(k)}$. Then each person i and his issue k play the same role in ${}^{\delta}_{\pi}P$ as person $\pi(i)$ and his issue $\delta(k)$ do in P.

Symmetric Linkage. There is $\lambda\colon M \to N$ in Λ such that for each permutation $\pi\colon N \to N$ and each permutation $\delta\colon M \to M$, if for each $i \in N$, δ maps the set of i's issues $\lambda^{-1}(i)$ *onto* the set of $\pi(i)$'s issues $\lambda^{-1}(\pi(i))$, then for each $k \in M$, $f_k\left({}^{\delta}_{\pi}P\right) = f_{\delta(k)}(P)$.

Next are two standard axioms of social choice, known as *anonymity* and *neutrality*. The former says that social decision should not depend on names of opinion holders and the latter says that social decision should not depend on how issues are labeled.

Anonymity. For each $P \in \mathcal{P}_{\mathrm{Tri}}$ and each permutation $\pi\colon N \to N$, $f({}_{\pi}P) = f(P)$, where ${}_{\pi}P \in \mathcal{P}_{\mathrm{Tri}}$ is such that for each $i \in N$ and each $k \in M$, ${}_{\pi}P_{ik} \equiv P_{\pi(i)k}$.

Neutrality. For each $P \in \mathcal{P}_{\mathrm{Tri}}$, each permutation $\delta\colon M \to M$, and each $k \in M$, $f_k\left({}^{\delta}P\right) = f_{\delta(k)}(P)$, where ${}^{\delta}P \in \mathcal{P}_{\mathrm{Tri}}$ is such that for each $i \in N$ and each $k \in M$, ${}^{\delta}P_{ik} \equiv P_{i\delta(k)}$.

Clearly, the combination of *anonymity* and *neutrality* implies *symmetric linkage* but the converse does not hold.

4 Preliminary Results

We distinguish powers into two types. The power on the k^{th} issue is (fully) *exclusive* if there is a person i who has the power on the k^{th} issue and no one else has the power on the k^{th} issue.

[10] In their model, Λ is the singleton containing the identify function and our axiom reduces to their "symmetry" axiom.

It is (fully) *non-exclusive* if all agents have the "equal" power on the k^{th} issue associated with a single consent-quotas function (or, on the dichotomous domain, a list of consent-quotas). We will show that the power on an issue is either exclusive or non-exclusive: see Remark 2. Thus either only one person has the power or all persons have the equal power.

Given a system of powers W, when the power on the k^{th} issue is non-exclusive, who has the power on this issue is not essential. Thus by changing $W_1(k)$, we may find other systems representing the same rule. Thus the following equivalence relation on systems of powers is natural. Two systems of powers W and W' are *equivalent*, denoted by $W \sim W'$, if for each k with $W_1(k) \neq W_1'(k)$, the power on the k^{th} issue is *non-exclusive* (so, $W_2(k) = W_2'(k)$). The following two extreme systems are notable. Under a *non-exclusive system of powers*, everyone has the non-exclusive power on every issue. Under a *monocentric system of powers*, one person has the exclusive power on every issue.

Lemma 1. *Assume that a rule f is represented by a system of powers W. Let $k \in M$, $i \equiv W_1(k)$, and $q(\cdot) \equiv W_2(k)$. Then for each $\nu \in \{1, \ldots, n\}$,* (i) *$q_+(\nu) + q_-(\nu) = \nu + 1$ and when $\nu \leq n-1$, $q_+(\nu) = q_0(\nu)$ if and only if for each $P \in \mathcal{P}_{Tri}$ with $||P^k_{+,-}|| = \nu$,*

$$f_k(P) = 1 \Leftrightarrow ||P^k_+|| \geq q_+(\nu). \tag{3}$$

(ii) $q(\nu) = (\nu+1, \nu+1, 1)$ *if and only if for each $P \in \mathcal{P}_{Tri}$ with $||P^k_{+,-}|| = \nu$, $f_k(P) = -1$.*
(iii) $q(\nu) = (1, 0, \nu+1)$ *if and only if for each $P \in \mathcal{P}_{Tri}$ with $||P^k_{+,-}|| = \nu$, $f_k(P) = 1$.*
Thus, if for each $\nu \in \{1, \ldots, n\}$, one of the three cases holds, then the power on the k^{th} issue is non-exclusive.

Proof. Let $\nu \in \{1, \ldots, n\}$. Assume $q_+(\nu) + q_-(\nu) = \nu + 1$ and $q_+(\nu) = q_0(\nu)$. Then the three parts (i)-(iii) in (2) collapse into (3). Conversely, if (3) holds, then from parts (ii) and (iii) in (2), $q_0(\nu) = q_+(\nu)$ and $q_+(\nu) = \nu + 1 - q_-(\nu)$.

Parts (ii) and (iii) are straightforward. Note that if any of the three cases (i)-(iii) holds, who has the power on the k^{th} issue is not essential. Changing $W_1(k)$ into any other person does not affect the rule the system represents, which means everyone has the power on the k^{th} issue associated with the same consent-quotas function. Thus the power is non-exclusive. ∎

We now show that the three cases of non-exclusive powers in Lemma 1 characterize non-exclusive powers.

Proposition 1. *The power on an issue associated with $(q_+(\cdot), q_0(\cdot), q_-(\cdot))$ is non-exclusive if and only if for each $\nu \in \{1, \ldots, n\}$,* (i) *$q_+(\nu) \leq \nu$, $q_-(\nu) \leq \nu$, $q_+(\nu) + q_-(\nu) = \nu + 1$, and when $\nu \leq n-1$, $q_0(\nu) = q_+(\nu)$, or* (ii) *$(q_+(\nu), q_-(\nu)) \in \{(\nu+1, 1), (1, \nu+1)\}$ and when $\nu \leq n-1$, $(q_+(\nu), q_0(\nu), q_-(\nu)) \in \{(\nu+1, \nu+1, 1), (1, 0, \nu+1)\}$.*

The proof is in Appendix A.1.

The next result is uniqueness of systems of powers representing a rule.

Proposition 2. *Assume $n \geq 4$. If a rule is represented by a system of powers, then the system is unique up to the equivalence relation $\sim$.*

The proof is in Appendix A.1.

We next state necessary and sufficient conditions on a system of powers which guarantee *monotonicity* or *symmetric linkage* of the rule the system represents.

A consent-quotas function $q(\cdot) \equiv (q_+(\cdot), q_0(\cdot), q_-(\cdot))$ has *component ladder property* if for each $\nu \in \{1, \ldots, n\}$, the following three inequalities hold whenever they are well-defined

$$\begin{array}{l} \text{(i) } q_+(\nu - 1) \leq q_+(\nu) \leq q_+(\nu - 1) + 1; \\ \text{(ii) } q_0(\nu - 1) \leq q_0(\nu) \leq q_0(\nu - 1) + 1; \\ \text{(iii) } q_-(\nu - 1) \leq q_-(\nu) \leq q_-(\nu - 1) + 1. \end{array} \tag{4}$$

The function has *intercomponent ladder property* if for each $\nu \in \{1, \ldots, n\}$,

$$q_+(\nu) \leq q_0(\nu - 1) + 1 \leq \nu - q_-(\nu) + 2. \tag{5}$$

The function has *ladder property* if it has the above two properties. We also say that a system of powers W has *ladder property* if its consent-quotas functions have ladder property. On the dichotomous domain, component ladder property has no bite and intercomponent ladder property reduces to $q_+ + q_- \leq n + 2$.

Proposition 3. *A rule represented by a system of powers satisfies monotonicity if and only if the system of powers has ladder property.*

The proof is given in Appendix A.2.

Symmetric linkage requires existence of a linkage "in Λ". Thus, the systems of powers in our main results in Section 5 are in

$$\mathfrak{W}^{\Lambda} \equiv \{W(\cdot) : W(\cdot) \text{ is a system of powers such that } W_1(\cdot) \in \Lambda\}.$$

The condition on the systems in $\mathfrak{W}^{\Lambda}$, necessary and sufficient for *symmetric linkage*, is *horizontal equality*: for each pair of persons i and $j \in N$ with the same number of issues under W_1, that is, $|W_1^{-1}(i)| = |W_1^{-1}(j)|$, their powers are associated with the same consent-quotas function, that is, for each $k \in W_1^{-1}(i)$ and each $l \in W_1^{-1}(j)$, $W_2(k) = W_2(l)$. When $i = j$, this property says that person i's powers on two different issues are associated with the same consent-quotas function.

Proposition 4. *A rule represented by a system of powers in $\mathfrak{W}^{\Lambda}$ satisfies symmetric linkage if and only if the system of powers satisfies horizontal equality.*

The proof is given in Appendix A.2.

5 Main Results

5.1 Monotonicity, Independence, and Symmetric Linkage

In this section, we state our results imposing *monotonicity*, *independence*, and *symmetric linkage*.

If a rule is represented by a system of powers, decisions on different issues are made independently and so the rule satisfies *independence*. By Propositions 3 and 4, if the system of powers is in $\mathfrak{W}^{\Lambda}$ and satisfies both ladder property and horizontal equality, the rule also satisfies *monotonicity* and *symmetric linkage*. Our first main result says that the converse also holds. That is, the combination of the three axioms is sufficient for existence of a system of powers in $\mathfrak{W}^{\Lambda}$.

Theorem 1. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$. A rule on $\mathcal{D}$ satisfies monotonicity, independence, and symmetric linkage if and only if it is represented by a system of powers in $\mathfrak{W}^{\Lambda}$ satisfying ladder property and horizontal equality. Moreover, the system is unique up to the equivalence relation $\sim$.*

The proof is in Appendix A.3. We show independence of the three axioms later.

Adding *anonymity*, we obtain:

Theorem 2. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$. The following are equivalent.* (i) *A rule on $\mathcal{D}$ satisfies monotonicity, independence, symmetric linkage, and anonymity.* (ii) *A rule on $\mathcal{D}$ is represented by a non-exclusive system of powers in $\mathfrak{W}^{\Lambda}$ satisfying ladder property and horizontal equality.* (iii) *A rule on $\mathcal{D}$ is represented by a system of powers $W \in \mathfrak{W}^{\Lambda}$ satisfying ladder property and horizontal equality such that for each $k \in M$, letting $(q_+(\cdot), q_0(\cdot), q_-(\cdot)) \equiv W_2(k)$, for each $\nu \in \{1, \ldots, n\}$,* (iii.1) *$q_+(\nu) \leq \nu$, $q_-(\nu) \leq \nu$, $q_+(\nu) + q_-(\nu) = \nu + 1$, and when $\nu \leq n-1$, $q_0(\nu) = q_+(\nu)$, or* (iii.2) *$(q_+(\nu), q_-(\nu)) \in \{(\nu+1, 1), (1, \nu+1)\}$ and when $\nu \leq n-1$, $(q_+(\nu), q_0(\nu), q_-(\nu)) \in \{(\nu+1, \nu+1, 1), (1, 0, \nu+1)\}$.*

Proof. Let $k \in M$ and $i \equiv W_1(k)$. By *anonymity*, when i has the power on the k^{th} issue, then every other agent should have the same power too. Thus the power on the k^{th} issue is non-exclusive. The proof for the reverse direction is straightforward. This proves the equivalence between (i) and (ii). We obtain the remaining equivalence from Proposition 1. ∎

On the dichotomous domain $\mathcal{P}_{\text{Di}}$, part (iii) of Theorem 2 can be simplified into the following: for each $k \in M$, letting $(q_+, q_-) \equiv W_2(k)$, (iii.1) $q_+ \leq n$, $q_- \leq n$ and $q_+ + q_- = n+1$ or (iii.2) $(q_+, q_-) \in \{(n+1, 1), (1, n+1)\}$.

Adding *neutrality* to the three axioms of Theorem 1, we characterize two extreme types of systems of powers, monocentric systems and non-exclusive systems.

Theorem 3. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$. A rule on $\mathcal{D}$ satisfies monotonicity, independence, symmetric linkage, and neutrality if and only if it is represented either by a monocentric system of powers in $\mathfrak{W}^{\Lambda}$ or by a constant non-exclusive system of powers in $\mathfrak{W}^{\Lambda}$ satisfying ladder property and horizontal equality, in either case.*[11]

Proof. If f is represented by a monocentric system of powers, then one and only one agent has the power on each issue. By horizontal equality, the consent-quotas functions for all issues are identical. Hence decisions on different issues are made neutrally. If f is represented by a constant non-exclusive system of powers, then because of the constancy and non-exclusiveness, f satisfies *neutrality*.

To prove the converse, let f be a rule satisfying the stated axioms. By Theorem 1, there is a system of powers $W \in \mathfrak{W}^{\Lambda}$ representing f. Suppose that there is $i \in N$ who has an exclusive power on the k^{th} issue. Then by *neutrality*, i should have the same exclusive power

[11] Here a *constant* system of powers means that the system is a constant function.

on every other issue. Thus, the system is monocentric. If there is no exclusive power, then by Proposition 2, the system is non-exclusive. And by *neutrality*, it is constant. ∎

We next consider *duality* (Samet and Schmeidler 2003). Each issue may be defined as representing a certain statement (a proposal) or its negation (the antiproposal): e.g. qualification or disqualification. Which representation is taken does not matter if the rule satisfies:

Duality. For each $P \in \mathcal{P}_{\text{Tri}}$, $f(-P) = -f(P)$.

On the trichotomous domain $\mathcal{P}_{\text{Tri}}$, *duality* is incompatible with the combination of the three axioms in Theorem 1. To show this, consider a rule f satisfying these three axioms. Then for each $i \in N$, each $k \in \lambda^{-1}(i)$, and each $P \in \mathcal{P}_{\text{Tri}}$ with $P_{ik} = 0$ and $||P_+^k|| = ||P_-^k||$, $f_k(-P) = f_k(P)$, violating *duality*. However, on the dichotomous domain $\mathcal{P}_{\text{Di}}$, adding *duality* allows us to pin down a smaller family of rules. A system of powers W has *quotas duality* if for each issue $k \in M$, the consent-quotas function $(q_+(\cdot), q_0(\cdot), q_-(\cdot)) \equiv W_2(k)$ satisfies $q_+(\cdot) = q_-(\cdot)$.

Theorem 4. *On the dichotomous domain $\mathcal{P}_{Di}$, a rule satisfies monotonicity, independence, symmetric linkage, and duality if and only if it is represented by a system of powers in $\mathfrak{W}^\Lambda$ satisfying ladder property, horizontal equality and quotas duality.*

Proof. Let f be a rule and $W \in \mathfrak{W}^\Lambda$ a system of powers of f such that for each $k \in M$, if we let $(q_+, q_-) \equiv W_2(k)$, $q_+ = q_-$. Let $i \in N$ and $k \in W_1^{-1}(i)$. Let $P \in \mathcal{P}_{\text{Di}}$. Note $(-P)_{ik} = -P_{ik}$, $||(-P)_+^k|| = ||P_-^k||$, and $||(-P)_-^k|| = ||P_+^k||$. Therefore, $||(-P)_-^k|| \geq q_- \Leftrightarrow ||P_+^k|| \geq q_+$ and $||(-P)_+^k|| \geq q_+ \Leftrightarrow ||P_-^k|| \geq q_-$. Then $f(-P) = -f(P)$. Hence f satisfies *duality*.

Conversely, let f be a rule satisfying the four axioms. By Theorem 1, there exists a system of powers $W \in \mathfrak{W}^\Lambda$ representing f. Let $k \in M$, $i \equiv W_1(k)$, and $(q_+, q_-) \equiv W_2(k)$. Suppose, by contradiction, that $q_+ \neq q_-$, say, $q_+ > q_-$ (the same argument applies when $q_+ < q_-$). Let r be the number such that $q_+ > r \geq q_-$. Then there exists $P \in \mathcal{P}_{\text{Di}}$ such that $P_{ik} = -1$ and $||P_-^k|| = r$. Then $f_k(P) = -1$. Since $(-P)_{ik} = 1$ and $||(-P)_+^k|| = ||P_-^k|| = r < q_+$, $f_k(-P) = -1$, contradicting *duality*. ∎

When n is even, no system of powers satisfies both part (iii.1) of Theorem 2 and quotas duality. However, when n is odd, the two properties imply majority rule. Thus we get:

Corollary 1. *Assume that n is odd. On the dichotomous domain $\mathcal{P}_{Di}$, majority rule is the only rule satisfying monotonicity, independence, symmetric linkage, anonymity, and duality.*

Replacing *anonymity* with *neutrality*, we get:

Corollary 2. *Assume that n is odd. On the dichotomous domain $\mathcal{P}_{Di}$, a rule satisfies monotonicity, independence, symmetric linkage, neutrality, and duality if and only if it is majority rule or is represented by a monocentric system of powers in $\mathfrak{W}^\Lambda$ satisfying ladder property, horizontal equality and quotas duality.*

Proof. To prove the nontrivial direction, let f be a rule satisfying the five axioms. Then by Theorem 3, it is represented by a monocentric system of powers or by a constant non-exclusive system of powers. In the former case, we are done. In the latter case, the rule satisfies *anonymity*. Thus it follows from Corollary 1 that f is majority rule. ∎

We now investigate consequences of dropping any one of *monotonicity*, *independence* and *symmetric linkage*, which shows independence of the three main axioms.

Dropping Symmetric Linkage

We characterize the following rules satisfying *monotonicity* and *independence*. These rules can be described by "decisive structures" between subgroups of N (Ju 2003).[12] Let $\mathfrak{C}^* \equiv \{(C_1, C_2) \in 2^N \times 2^N : C_1 \cap C_2 = \varnothing\}$ be the set of all pairs of disjoint subgroups of N. For each $k \in M$, a *decisive structure for the k^{th}-issue*, denoted by $\mathfrak{C}_k \subseteq \mathfrak{C}^*$, is a subset of $\mathfrak{C}^*$. It satisfies *monotonicity* if for each $(C_1, C_2) \in \mathfrak{C}_k$, if $(C_1', C_2') \in \mathfrak{C}^*$ is such that $C_1' \supseteq C_1$ and $C_2' \subseteq C_2$, then $(C_1', C_2') \in \mathfrak{C}_k$. For each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$, let

$$N\left(P_+^k\right) \equiv \{i \in N : P_{ik} = 1\} \text{ and } N\left(P_-^k\right) \equiv \{i \in N : P_{ik} = -1\}.$$

A rule f is *represented by a profile of decisive structures* $(\mathfrak{C}_k)_{k \in M}$ if for each $P \in \mathcal{D}$ and each $k \in M$, $f_k(P) = 1$ if and only if $(N(P_+^k), N(P_-^k)) \in \mathfrak{C}_k$ (thus, when $(C_1, C_2) \in \mathfrak{C}_k$, unanimously positive opinions on the k^{th} issue among the members of C_1 can overrule unanimously negative opinions among the members of C_2). Any rule represented by a profile of decisive structures satisfies *independence*, since it makes decisions issue by issue. Conversely, if a rule satisfies *independence*, the decision on the k^{th} issue relies only on the pair of the set of persons in favor of k and the set of persons against k. Thus, it is represented by a profile of decisive structures. Monotonicity of decisive structures is a necessary and sufficient condition for *monotonicity* of the rule. Therefore we obtain:

Proposition 5. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$.* (i) *A rule on $\mathcal{D}$ satisfies independence if and only if it is represented by a profile of decisive structures.* (ii) *A rule on $\mathcal{D}$ satisfies independence and monotonicity if and only if it is represented by a profile of monotonic decisive structures.*

The formal proof is left for readers.

Let $\mathcal{I}^* \equiv \{(n_1, n_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : n_1 + n_2 \leq n\}$, where $\mathbb{Z}_+$ is the set of non-negative integers. Any subset $\mathcal{I} \subseteq \mathcal{I}^*$ is called an *index set*. It is *comprehensive* if for each $(n_1, n_2) \in \mathcal{I}$ and each $(n_1', n_2') \in \mathcal{I}^*$, if $n_1' \geq n_1$ and $n_2' \leq n_2$, then $(n_1', n_2') \in \mathcal{I}$. Using Proposition 5, it is easy to characterize rules satisfying *independence* and *anonymity*. Decisive structures of each of these rules can be described by index sets. Formally, a *counting rule* is a rule that is represented by a profile of index sets, $(\mathcal{I}_k)_{k \in M}$, as follows: for each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$, $f_k(P) = 1 \Leftrightarrow (||P_+^k||, ||P_-^k||) \in \mathcal{I}_k$ (thus, $(n_1, n_2) \in \mathcal{I}_k$ means that n_1 positive opinions on the k^{th} issue can overrule n_2 negative opinions on the k^{th} issue). It is easy to show that a counting rule is *monotonic* if and only if all index sets in the profile $(\mathcal{I}_k)_{k \in M}$ are comprehensive. Thus, we obtain:

Proposition 6. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$.* (i) *A rule on $\mathcal{D}$ satisfies independence and anonymity if and only if it is a counting rule.* (ii) *A rule on $\mathcal{D}$ satisfies monotonicity, independence,*

[12] Ju (2003) calls decisive structures "power structures". We use different name to avoid confusion with our stronger notion of power.

and anonymity if and only if it is a counting rule represented by a profile of comprehensive index sets.

The formal proof is left for readers.

Dropping Monotonicity

An *extended system of powers* ${}_eW$ maps each issue $k \in M$ into a person ${}_eW_1(k) \in N$ and a triple of index sets ${}_eW_2(k) = (\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k)$. A rule f is *represented by an extended system of powers* ${}_eW$ if for each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$,

$$\begin{array}{l} \text{(i) when } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow \left(||P_+^k||, ||P_-^k||\right) \in \mathcal{I}_+^k; \\ \text{(ii) when } P_{ik} = 0,\ f_k(P) = 1 \Leftrightarrow \left(||P_+^k||, ||P_-^k||\right) \in \mathcal{I}_0^k; \\ \text{(iii) when } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow \left(||P_-^k||, ||P_+^k||\right) \in \mathcal{I}_-^k; \end{array} \tag{6}$$

where $i \equiv {}_eW_1(k)$ and $(\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k) \equiv {}_eW_2(k)$. Note that index sets are used for this rule in a different manner from how they are used for a counting rule. To each issue $k \in M$, three index sets correspond and they describe when the person with the power on issue k gets a sufficient consent from the society. A rule represented by an extended system of powers is not necessarily a counting rule. *Symmetric linkage* together with *independence* force us to have extended systems in the following set:

$${}_e\mathfrak{W}^\Lambda \equiv \{{}_eW(\cdot) : {}_eW(\cdot) \text{ is an extended system of powers such that } {}_eW_1(\cdot) \in \Lambda\}.$$

Proposition 7. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$. A rule on $\mathcal{D}$ satisfies independence and symmetric linkage if and only if it is represented by an extended system of power ${}_eW(\cdot) \in {}_e\mathfrak{W}^\Lambda$ satisfying horizontal equality, that is, for each $i, j \in N$ with $|{}_eW_1^{-1}(i)| = |{}_eW_1^{-1}(j)|$, each $k \in {}_eW_1^{-1}(i)$, and each $l \in {}_eW_1^{-1}(j)$, ${}_eW_2(k) = {}_eW_2(l)$.*[13]

The proof is in Appendix A.3.

Dropping Independence

For each $P \in \mathcal{P}_{\text{Tri}}$, let $\chi(P) \equiv \sum_{k \in M} ||P_-^k||/|M|$. Let f be the rule represented by $\chi(\cdot)$ as follows: for each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$, $f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq \chi(P)$. By definition, this rule treats agents anonymously and issues neutrally. Thus it satisfies *anonymity*, *neutrality*, and so *symmetric linkage*. If $P, P' \in \mathcal{P}_{\text{Tri}}$ are such that for each $k \in M$, $N\left(P_+^k\right) \subseteq N\left(P_+'^k\right)$ and $N\left(P_-^k\right) \supseteq N\left(P_-'^k\right)$, $\sum_{k \in M} ||P_-^k||/|M| \geq \sum_{k \in M} ||P_-'^k||/|M|$, that is, $\chi(P) \geq \chi(P')$. Then for each $k \in M$, if $f_k(P) = 1$ (that is, $||P_+^k|| \geq \chi(P)$), $||P_+'^k|| \geq ||P_+^k|| \geq \chi(P) \geq \chi(P')$ and so $f_k(P') = 1$. Thus f satisfies *monotonicity*. The threshold level $\chi(P)$ depends on opinions on all issues. So f violates *independence*. Using different $\chi(\cdot)$, we can define other examples of rules violating *independence* but satisfying other axioms. However, we leave it for future research to characterize the entire family of rules satisfying *monotonicity* and *symmetric linkage*.

[13] This property of ${}_eW(\cdot)$ is needed to guarantee *symmetric linkage* like horizontal equality of a system of powers.

Anonymity and Representation by A Non-Exclusive System of Powers

Consider a rule f satisfying *monotonicity* and *independence*. If f also satisfies *anonymity*, then f is a monotonic counting rule. Thus, there is a profile of comprehensive index sets $(\mathcal{I}_k)_{k \in M}$ representing f. For each $\nu \in \{0, 1, \ldots, n\}$, if $\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}_k\} \neq \emptyset$, let $q_+^k(\nu) \equiv \min\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}_k\}$; otherwise, $q_+^k(\nu) \equiv \nu + 1$, and let $q_0^k(\nu) \equiv q_+^k(\nu)$ and $q_-^k(\nu) \equiv \nu + 1 - q_+^k(\nu)$. Then we obtain one of the three cases in Lemma 1 and so f is represented by a non-exclusive system. Conversely, if f is represented by a non-exclusive system, f satisfies *anonymity*. Therefore, we obtain:

Proposition 8. *Let f be a rule satisfying monotonicity and independence. Then the following are equivalent.* (i) *Rule f satisfies* anonymity. (ii) *Rule f is represented by a non-exclusive system of powers satisfying ladder property.* (iii) *Rule f is a monotonic counting rule.*

If a monotonic counting rule f has at most n different index sets, then f can be represented by a system of powers satisfying horizontal equality. This is because there is a function $W_1(\cdot)$ that maps each pair $k, l \in M$ with the same index set $\mathcal{I}$ into the same person. Thus f satisfies *symmetric linkage*. If f has more than n different index sets, f violates *symmetric linkage*. Thus we obtain:

Corollary 3. *Assume $n \geq m$. Let f be a rule satisfying monotonicity and independence. Then the following are equivalent.* (i) *Rule f satisfies* anonymity. (ii) *Rule f is represented by a non-exclusive system of powers satisfying ladder property and horizontal equality.* (iii) *Rule f is a monotonic counting rule.*

Remark 1. This proposition shows that if $n \geq m$, *monotonicity*, *independence* and *anonymity* together imply *symmetric linkage*. Therefore, in this case, *symmetric linkage* in Theorem 2 and Corollary 1 can be dropped.

Models with A Unique Linkage between Issues and Persons

Consider the model with a unique linkage in Λ, denoted by $\lambda \colon M \to N$: e.g. $M = N$ and λ is the identity function as in Samet and Schmeidler (2003). Then the earlier results in this section give characterizations of rules represented by systems of powers $W(\cdot)$ *conforming to the unique linkage*, that is, $W_1(\cdot) = \lambda(\cdot)$. Depending on $\lambda(\cdot)$, some of our results can be strengthened. For example, suppose that $\lambda(\cdot)$ is not constant. Then no system of powers conforming to $\lambda(\cdot)$ can be monocentric. Thus, it follows from Theorem 3 that:

Corollary 4. *Assume that there is a unique linkage $\lambda \colon M \to N$ in Λ and λ is not constant. A rule over $\mathcal{D} \in \{\mathcal{P}_{Tri}, \mathcal{P}_{Di}\}$ satisfies monotonicity, independence, symmetric linkage, and neutrality if and only if it is represented by a constant non-exclusive system of powers conforming to $\lambda(\cdot)$ and satisfying ladder property and horizontal equality. Thus the four axioms together imply* anonymity.

Also it follows from Corollary 2 that:

Corollary 5. *Assume that there is a unique linkage $\lambda\colon M \to N$ in Λ and λ is not constant. When n is odd, majority rule is the only rule on $\mathcal{P}_{Di}$ satisfying monotonicity, independence, symmetric linkage, neutrality, and duality.*

5.2 Pareto Efficiency and Existence of A System of Powers

Compatibility of *Pareto efficiency* and existence of so-called libertarian rights (decisive powers) is widely studied by a number of authors followed by the celebrated work, Sen (1970). To discuss this issue in our framework, we now consider preference relations.

Opinions are partial description of the following preference relations. A *separable preference relation* R_0 orders social decisions in such a way that for each $k \in M$ and each quadruple $x, x', y, y' \in \{-1, 1\}^M$ with $x_k = y_k$, $x'_k = y'_k$, $x_{-k} = x'_{-k}$, and $y_{-k} = y'_{-k}$, we have $x \succ_{R_0} x' \Leftrightarrow y \succ_{R_0} y'$ and $x \sim_{R_0} x' \Leftrightarrow y \sim_{R_0} y'$, where $\succ_{R_0}$ and $\sim_{R_0}$ are strict and indifference relations associated with R_0. Then issues are partitioned into goods, bads, and nulls depending on whether they have positive or negative or indifferent impacts on the person's well-being. Thus, each separable preference R_0 is associated with an opinion vector P_0, each positive (resp. negative or zero) component of P_0 representing the corresponding issue as a good (resp. a bad or a null). Obviously, there are a number of separable preference relations corresponding to a single opinion vector. Let $\mathcal{R}$ be the family of profiles of separable preference relations. A *rule* on the separable preferences domain $\mathcal{R}$ associates with each profile of preference relations a single alternative in $\{-1, 1\}^M$. With the above stated relationship between opinions and preferences, axioms and powers defined for the opinion domain are easily extended to the corresponding notions on the separable preferences domain.

5.2.1 Sen's Paradox of Paretian Liberal

Sen (1970) shows in the Arrovian social choice model that there is no *Pareto efficient* preference aggregation rule that gives at least two agents libertarian rights. This is so-called Sen's paradox of Paretian liberal. Sen's reasoning does not apply directly in our model as we focus on separable preference relations and consider social choice functions instead of preference aggregation rules in Sen (1970). Yet, our notion of decisive powers is a natural counterpart to Sen's libertarian rights as is noticed by Gibbard (1974).[14] Thus Sen's quest is still meaningful here. Does Sen's paradox hold in our model? Not surprisingly, it does, as we show below. Furthermore, we show that the paradox holds in a much stronger sense.

We first show that the paradox holds on the separable preferences domain.[15] Sen's (1970) *minimal liberalism* postulates that there should be at least two persons who have decisive powers. Assume that persons 1 and 2 are given the decisive powers on the first issue and the second issue respectively. Consider the following preference relations of the two persons. For person 1, the first issue is a bad and the second issue is a good. But person 1 cares so much about the second issue (person 2's issue) that the positive decision on this issue is preferred to the negative decision no matter what decisions are made on the other issues. For

[14]As Sen (1983, p.14) points out, the so-called Gibbard paradox does not hold on the domain of separable preference relations.

[15]This was originally proven by Gibbard (1974, Theorem 2).

person 2, the second issue is a bad and the first issue is a good. But person 2 cares so much about the first issue (person 1's issue) that the positive decision on this issue is preferred to the negative decision no matter what decisions are made on the other issues. Then by the decisive powers of the two persons, decisions on the first and second issues are both negative. But the two persons will be better off at any decision with positive components for both issues. This confirms that *minimal liberalism and Pareto efficiency are incompatible on the separable preferences domain.*

Preference relations in the above example are "meddlesome"' (Blau 1975). One may hope that without such meddlesome preference relations, the paradox of Paretian liberal will not apply. Unfortunately, the paradox holds even in a substantially restricted environment where only trichotomous or dichotomous preference relations are admissible. A *trichotomous preference relation* R_0 is a separable preference relation represented by a function $U_0\colon \{-1,1\}^M \to \mathbb{R}$ such that for each $x \in \{-1,1\}^M$, $U_0(x) = \sum_{k\in M: x_k=1} P_{0k}$, where $P_0 \in \{-1,0,1\}^M$ is the opinion vector corresponding to R_0.[16] A *dichotomous preference relation* is a trichotomous preference relation for which each issue is either a good or a bad. Let $\mathcal{R}_{\mathrm{Tri}}$ be the family of profiles of trichotomous preference relations and $\mathcal{R}_{\mathrm{Di}}$ the family of profiles of dichotomous preference relations. Note that there are one-to-one correspondences between $\mathcal{R}_{\mathrm{Tri}}$ and $\mathcal{P}_{\mathrm{Tri}}$ and between $\mathcal{R}_{\mathrm{Di}}$ and $\mathcal{P}_{\mathrm{Di}}$.

Proposition 9. *When there are at least three persons, no Pareto efficient rule on the dichotomous* (*or trichotomous*) *preferences domain satisfies minimal liberalism.*

Proof. Suppose that persons 1 and 2 have the decisive powers respectively on issue 1 and issue 2. Consider the profile of dichotomous preference relations $(R_i)_{i\in N}$ given by the following opinion vectors: $P_1 \equiv (1,-1,-1,\ldots,-1)$, $P_2 \equiv (-1,1,-1,\ldots,-1)$, and for each $i \in N\backslash\{1,2\}$, $P_i \equiv (-1,\ldots,-1)$. Then by the decisive powers of persons 1 and 2, $f_1(R) = f_2(R) = 1$. If the rule is *Pareto efficient*, for each $k \in M\backslash\{1,2\}$, $f_k(R) = -1$. Thus $f(R) = (1,1,-1,\ldots,-1)$. Note that this alternative is indifferent to $x \equiv (-1,\ldots,-1)$ for both person 1 and person 2 and x is preferred to $f(R)$ by all others. This contradicts *Pareto efficiency*. ∎

Note that unlike the previous paradox on the separable preferences domain, we need the assumption $n \geq 3$. The case with two persons ruled out by this assumption is very limited. However, it should be noted that the paradox does not apply in the two-person case (then decisiveness is quite close to plurality principle since one person's opinion accounts for 50%). This is an implication of our results in the next section.

5.2.2 Quasi-Plurality Systems of Powers

The observations made in Section 5.2.1 show that decisiveness component in the definition of libertarian rights is too strong to be compatible with *Pareto efficiency*. A way to escape from this impossibility is to weaken the decisiveness. Is it, then, possible to have non-decisive powers and at the same time to satisfy *Pareto efficiency*? It is indeed possible on the trichotomous preferences domain $\mathcal{R}_{\mathrm{Tri}}$ and also on the dichotomous preferences domain

[16] That is, $U_0(x) = |\{k \in M : x_k = 1 \text{ and } P_{0k} = 1\}| - |\{k \in M : x_k = 1 \text{ and } P_{0k} = -1\}|$.

$\mathcal{R}_{\mathrm{Di}}$ as we show in this section. Moreover, we provide a characterization of plurality-like rules on the basis of *Pareto efficiency*, *independence*, and *symmetric linkage*. Since we only consider trichotomous or dichotomous preference relations, throughout this section, we use opinion vectors to refer to the corresponding trichotomous preference relations.

We begin with a definition of important systems of powers in the current section.

Definition 2 (Quasi-Plurality Systems of Powers)**.** A system of powers W is called a *quasi-plurality system* if there is a consent-quotas function $q(\cdot) \equiv (q_+(\cdot), q_0(\cdot), q_-(\cdot))$ such that for each $k \in M$, $W_2(k) = (q_+(\cdot), q_0(\cdot), q_-(\cdot))$ and for each $\nu \in \{1, \ldots, n\}$,

$$q_+(\nu), q_-(\nu) \in \{\frac{\nu-1}{2}, \frac{\nu+1}{2}\}, \tag{7}$$

for each $\nu \in \{0, \ldots, n-1\}$,

$$q_0(\nu) \in \{\frac{\nu-1}{2}, \frac{\nu+1}{2}\}. \tag{8}$$

The rule represented by a quasi-plurality system is called a *quasi-plurality rule*.

Clearly, quasi-plurality systems satisfy horizontal equality and thus quasi-plurality rules satisfy *symmetric linkage* if their representing systems of powers are in $\mathfrak{W}^{\Lambda}$. Obviously, plurality rule is an example; it is represented by a non-exclusive quasi-plurality system. There are also exclusive quasi-plurality systems. For example, for each $\nu \in \{1, \ldots, n\}$, let $q_+(\nu) = q_-(\nu) \equiv (\nu-1)/2$ and for each $\nu \in \{0, \ldots, n-1\}$, let $q_0(\nu) \equiv (\nu-1)/2$. Then the power on each issue is exclusive by Proposition 1. However, note that for each $k \in M$, if $||P_+^k|| \neq ||P_-^k||$, $f_k(P)$ equals the decision made by plurality rule and that if $||P_+^k|| = ||P_-^k||$, $f_k(P)$ is determined by the opinion of the person, say i, who has the power on the k^{th} issue (that is, $f_k(P) = 1$ if $P_{ik} = 1$ or 0; $f_k(P) = -1$ if $P_{ik} = -1$). Thus "exclusiveness" feature, if it exists, plays only a tie-breaking role when the group of persons with the positive opinion and the group of persons with the negative opinion have the same size.

Any quasi-plurality rule f has the following property: for each $k \in M$,

$$\text{if } f_k(P) = 1, ||P_+^k|| \geq ||P_-^k|| \text{ ; if } ||P_+^k|| > ||P_-^k||, f_k(P) = 1. \tag{9}$$

Note that $\sum_{i \in N} U_i(x) = \sum_{i \in N} \sum_{\{k \in M : x_k = 1\}} P_{ik} = \sum_{\{k \in M : x_k = 1\}} (||P_+^k|| - ||P_-^k||)$. Therefore, by (9), any quasi-plurality rule maximizes the sum of utilities. Thus it satisfies *Pareto efficiency*. Moreover, our next result shows that quasi-plurality rules are the only rules satisfying *Pareto efficiency*, *independence*, and *symmetric linkage*.

Theorem 5. *Assume that each linkage λ: $M \to N$ in Λ maps to all persons the same number of issues. Then a rule on $\mathcal{D} \in \{\mathcal{R}_{Tri}, \mathcal{R}_{Di}\}$ satisfies* Pareto efficiency, independence, *and* symmetric linkage *if and only if it is represented by a quasi-plurality system of powers conforming to λ.*[17]

The proof is in Appendix A.4. Note that this result holds in Samet and Schmei-

[17]The characterization of "semi-plurality rules" by Ju (2005) imposes *anonymity* instead of *symmetric linkage*. Note also that this result holds only on $\mathcal{R}_{\mathrm{Tri}}$, while our Theorem 5 holds both on $\mathcal{R}_{\mathrm{Tri}}$ and on $\mathcal{R}_{\mathrm{Di}}$. The family of quasi-plurality rules is larger than the family of semi-plurality rules in Ju (2005).

dler's (2003) model because in their model $N = M$ and the identity function is the only linkage in Λ. Not all quasi-plurality systems satisfy intercomponent ladder property. This extra property is obtained after adding *monotonicity* to the three axioms in the theorem.

Next is a direct corollary to Theorem 5.

Corollary 6. *Given the assumption in Theorem 5, a rule on $\mathcal{D} \in \{\mathcal{R}_{Tri}, \mathcal{R}_{Di}\}$, represented by a system of powers conforming to the unique linkage λ, satisfies* Pareto efficiency *if and only if the system of powers is a quasi-plurality system.*

When the number of issues is greater than or equal to the number of persons, adding *neutrality*, we establish the same characterization without any assumption on Λ.

Theorem 6. *Suppose $m \geq n$. A rule on $\mathcal{D} \in \{\mathcal{R}_{Tri}, \mathcal{R}_{Di}\}$ satisfies* Pareto efficiency, independence, symmetric linkage, *and* neutrality *if and only if it is represented either by a non-exclusive quasi-plurality system of powers in $\mathfrak{W}^{\Lambda}$ or by a monocentric quasi-plurality system of powers in $\mathfrak{W}^{\Lambda}$.*

The proof is in Appendix A.4.

A Proofs

A.1 Proofs of Propositions 1 and 2

Let $\mathfrak{W}_f$ be the set of systems of powers representing a rule f.

Claim 1. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W_1'(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W_2'(k)$. Then for each $\nu \geq 2$,* (i) *if $q_+(\nu) \geq 2$ and $q_+'(\nu) \geq 2$, $q_+(\nu) = q_+'(\nu)$;* (ii) *if $q_-(\nu) \geq 2$ and $q_-'(\nu) \geq 2$, $q_-(\nu) = q_-'(\nu)$.*

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$. We prove (i) and skip the same proof of (ii). Suppose that $q_+(\nu) \neq \nu + 1$ and $q_+'(\nu) \neq \nu + 1$. Because $\nu \geq 2$, $q_+(\nu) \neq \nu + 1$, and $q_+(\nu) \geq 2$, there exists P such that $||P^k_{+,-}|| = \nu$, $P_{ik} = P_{i'k} = 1$ and $||P^k_+|| = q_+(\nu)$. Then by i's power $W(k)$, $f_k(P) = 1$. Thus, by i''s power $W'(k)$, $q_+'(\nu) \leq q_+(\nu)$. Similarly, we show the reverse inequality. If $q_+(\nu) = \nu + 1$, then consider P such that $||P^k_{+,-}|| = \nu$, $P_{ik} = P_{i'k} = 1$ and $||P^k_+|| = \nu$. By i's power $W(k)$, $f_k(P) = -1$. Then by i''s power $W'(k)$, $q_+'(\nu) > ||P^k_+|| = \nu$, which implies $q_+'(\nu) = \nu + 1$. ∎

Claim 2. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W_1'(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W_2'(k)$. Then for each $\nu \geq 2$,* (i) *$q_+(\nu) = 1$ if and only if $q_-'(\nu) \geq \nu$;* (ii) *$q_-(\nu) = 1$ if and only if $q_+'(\nu) \geq \nu$.*

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$. We prove the first equivalence and skip the same proof of the second. Assume $q_+(\nu) = 1$. Since $\nu \geq 2$, there is P such that $||P^k_{+,-}|| = \nu$, $P_{ik} = 1$, $P_{i'k} = -1$, and $||P^k_+|| = 1$ (so $||P^k_-|| = \nu - 1$). Then by i's power $W(k)$, $f_k(P) = 1$. Hence by i''s power $W'(k)$, $q_-'(\nu) > \nu - 1$, that is, $q_-'(\nu) \geq \nu$. Conversely, if $q_-'(\nu) \geq \nu$, then using the same P as above and i''s power $W'(k)$, we have $f_k(P) = 1$. This means, by i's power $W(k)$, $q_+(\nu) \leq 1$. Thus $q_+(\nu) = 1$. ∎

Claim 3. *Assume that* $W, W' \in \mathfrak{W}_f$ *and for some* $k \in M$, $W_1(k) \neq W_1'(k)$. *Let* $q(\cdot) \equiv W_2(k)$ *and* $q'(\cdot) \equiv W_2'(k)$. *Then for each* $\nu \in \{1, \ldots, n-1\}$, (i) *if* $q_0(\nu) \geq 1$, $q_0(\nu) = q_+'(\nu)$; (ii) $q_0(\nu) = 0$ *if and only if* $q_+'(\nu) = 1$ *and* $q_-'(\nu) = \nu + 1$.

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$.

Part (i). Suppose $q_0(\nu) \geq 1$. If $q_0(\nu) = \nu + 1$, then for each P with $||P_{+,-}^k|| = \nu$, $P_{ik} = 0$ and $P_{i'k} = 1$, by i's power $W(k)$, $f_k(P) = -1$. Thus by i''s power $W'(k)$, $q_+'(\nu) = \nu + 1$. If $q_0(\nu) \leq \nu$, there exists P such that $||P_{+,-}^k|| = \nu$, $P_{ik} = 0$, $P_{i'k} = 1$, and $||P_+^k|| = q_0(\nu)$ (such P exists because $1 \leq q_0(\nu) \leq \nu \leq n-1$). Then by $W(k)$, $f_k(P) = 1$. And by $W'(k)$, $q_+'(\nu) \leq q_0(\nu)$. Thus if $q_0(\nu) = 1$, $q_+'(\nu) = 1$. Suppose $q_0(\nu) \geq 2$. In this case, if $q_+'(\nu) < q_0(\nu)$, there exists P with $||P_{+,-}^k|| = \nu$, $P_{ik} = 0$, $P_{i'k} = 1$, and $q_+'(\nu) \leq ||P_+^k|| < q_0(\nu)$ (such P exists because $q_0(\nu) \geq 2$). Then by $W'(k)$, $f_k(P) = 1$ and by $W(k)$, $f_k(P) = -1$, which is a contradiction.

Part (ii). Suppose $q_0(\nu) = 0$. Consider P such that $||P_{+,-}^k|| = \nu$, $P_{ik} = 0$, $P_{i'k} = 1$, and $||P_+^k|| = 1$. By $W(k)$, $f_k(P) = 1$. Then by $W'(k)$, $q_+'(\nu) \leq 1$. Thus $q_+'(\nu) = 1$. Next consider P such that $||P_{+,-}^k|| = \nu$, $P_{ik} = 0$, $P_{i'k} = -1$, and $||P_-^k|| = \nu$. By $W(k)$, $f_k(P) = 1$. Then by $W'(k)$, $q_-'(\nu) > \nu$. Thus $q_-'(\nu) = \nu + 1$. To prove the converse, suppose $q_+'(\nu) = 1$ and $q_-'(\nu) = \nu + 1$. If P is such that $||P_{+,-}^k|| = \nu$, $P_{ik} = 0$, $P_{i'k} = -1$, and $||P_-^k|| = \nu$, then by $W'(k)$, $f_k(P) = 1$. Thus by $W(k)$, $q_0(\nu) \leq 0$. ∎

Claim 4. *Assume that* $W, W' \in \mathfrak{W}_f$ *and for some* $k \in M$, $W_1(k) \neq W_1'(k)$. *Let* $q(\cdot) \equiv W_2(k)$ *and* $q'(\cdot) \equiv W_2'(k)$. *Then for each* $\nu \in \{3, \ldots, n\}$, $q_+(\nu) = q_+'(\nu)$ *and* $q_-(\nu) = q_-'(\nu)$.

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$. We first show $q_+(\nu) = q_+'(\nu)$. If both numbers are greater than or equal to 2, the result follows from Claim 1. Suppose $q_+(\nu) = 1$. Then by Claim 2, $q_-'(\nu) \geq \nu$. If $q_-(\nu) \neq q_-'(\nu) (\geq \nu \geq 3)$, then $q_-(\nu) = 1$ (because otherwise, by Claim 1, $q_-(\nu) = q_-'(\nu)$). Let P be such that $||P_{+,-}^k|| = \nu$, $P_{ik} = P_{i'k} = -1$ and $||P_-^k|| = 2$. Since $q_-(\nu) = 1 < ||P_-^k|| < 3 \leq \nu \leq q_-'(\nu)$, then by $W(k)$, $f_k(P) = -1$ and by $W'(k)$, $f_k(P) = 1$, which is a contradiction. Therefore $q_-(\nu) = q_-'(\nu) \geq \nu$. Then by Claim 2, $q_+'(\nu) = 1$.

We next show $q_-(\nu) = q_-'(\nu)$. If both numbers are greater than or equal to 2, the result follows from Claim 1. Suppose $q_-(\nu) = 1$. Then by Claim 2, $q_+'(\nu) \geq \nu$. Since $q_+(\nu) = q_+'(\nu) \geq \nu$, by Claim 2 again, $q_-'(\nu) = 1$. ∎

Claim 5. *Assume that* $W, W' \in \mathfrak{W}_f$ *and for some* $k \in M$, $W_1(k) \neq W_1'(k)$. *Let* $q(\cdot) \equiv W_2(k)$ *and* $q'(\cdot) \equiv W_2'(k)$. *Then for each* $\nu \in \{0, \ldots, n-2\}$, $q_0(\nu) = q_0'(\nu)$ *and when* $n \geq 4$, $q_0(n-1) = q_0'(n-1)$.

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$. Let $\nu \in \{0, \ldots, n-2\}$. Suppose $q_0(\nu) \neq q_0'(\nu)$, say, $q_0(\nu) < q_0'(\nu)$. Since $\nu \leq n-2$ and $q_0(\nu) \leq \nu$ (note $q_0(\nu) < q_0'(\nu) \leq \nu + 1$), then there is P be such that $P_{ik} = P_{i'k} = 0$, $||P_{+,-}^k|| = \nu$, and $||P_+^k|| = q_0(\nu)$. Then by $W(k)$, $f_k(P) = 1$ and by $W'(k)$, $f_k(P) = -1$, which is a contradiction.

Finally, $q_0(n-1) = q_0'(n-1)$ follows from Claim 3 and the fact that $q_+(n-1) = q_+'(n-1)$ and $q_-(n-1) = q_-'(n-1)$, which holds by Claim 4 (here we need the assumption of $n \geq 4$ in order to have $n - 1 \geq 3$). ∎

Claim 6. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W'_1(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W'_2(k)$. Then for each $\nu \in \{1, \ldots, n-1\}$, $q_-(\nu) = 1$ if and only if $q'_0(\nu) \geq \nu$.*

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W'_1(k)$. Let $\nu \in \{1, \ldots, n-1\}$. Suppose $q_-(\nu) = 1$. Let P be such that $||P^k_{+,-}|| = \nu$, $P_{ik} = -1$, $P_{i'k} = 0$, and $||P^k_+|| = \nu - 1$ (so $||P^k_-|| = 1$). By $W(k)$, $f_k(P) = -1$. Thus by $W'(k)$, $q'_0(\nu) > \nu - 1$, that is, $q'_0(\nu) \geq \nu$. Suppose $q'_0(\nu) \geq \nu$. Consider the same P as above. By $W'(k)$, $f_k(P) = -1$. Thus by $W(k)$, $q_-(\nu) \leq 1$ and so $q_-(\nu) = 1$. ∎

Lemma 2. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W'_1(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W'_2(k)$. Then for each $\nu \in \{0, 1, \ldots, n\}$, if $\nu \geq 1$, $q_+(\nu) = q'_+(\nu)$ and $q_-(\nu) = q'_-(\nu)$; if $\nu \leq n-1$, $q_0(\nu) = q'_0(\nu)$.*

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W'_1(k)$. By Claims 4 and 5, we only need to show that for each $\nu \in \{1, 2\}$, $q_+(\nu) = q'_+(\nu)$ and $q_-(\nu) = q'_-(\nu)$.

Consider $\nu = 2$. Then $q_0(\nu) = q'_0(\nu)$ by Claim 5. If $q_0(\nu) = 0$, then by Claim 3, $q'_+(\nu) = 1 = q_+(\nu)$. If $q_0(\nu) = q'_0(\nu) \geq 1$, then applying (i) of Claim 3 twice, $q'_0(\nu) = q_+(\nu)$ and $q_0(\nu) = q'_+(\nu)$. Thus $q_+(\nu) = q'_+(\nu)$. We next show $q_-(\nu) = q'_-(\nu)$. If both numbers are greater than or equal to 2, the result follows from Claim 1. Suppose $q_-(\nu) = 1$. Then by Claim 2, $q'_+(\nu) \geq \nu$. Since $q_+(\nu) = q'_+(\nu) \geq \nu$, then by Claim 2 again, $q'_-(\nu) = 1$.

Now consider $\nu = 1$. By Claim 5, $q_0(1) = q'_0(1)$. Suppose $q_0(1) = q'_0(1) \geq 1$. Then by Claim 3, $q_+(1) = q'_+(1)$. And by Claim 6, $q_-(1) = q'_-(1)$. Suppose $q_0(1) = q'_0(1) = 0$. Then by Claim 3, $q_+(1) = q'_+(1) = 1$ and $q_-(1) = q'_-(1) = 2$. ∎

Claim 7. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W'_1(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W'_2(k)$. Then for each $\nu \in \{1, \ldots, n\}$, $q_+(\nu) + q'_-(\nu) > \nu$ (and $q'_+(\nu) + q_-(\nu) > \nu$).*

Proof. The inequalities hold trivially for $\nu = 1$. Let $\nu \in \{2, \ldots, n\}$. Let $i \equiv W_1(k)$ and $i' \equiv W'_1(k)$. Suppose by contradiction $q_+(\nu) + q'_-(\nu) \leq \nu$. Then $q_+(\nu) < \nu$ or $q'_-(\nu) < \nu$. In the former case (we skip the same proof for the latter case). There is $P \in \mathcal{P}_{\text{Tri}}$ such that $P_{ik} = 1$, $P_{i'k} = -1$, $||P^k_{+,-}|| = \nu$, and $||P^k_+|| = q_+(\nu)$ (such P exists because $\nu \geq 2$, $q_+(\nu) < \nu$, and so $||P^k_-|| = \nu - q_+(\nu) \geq 1$). Then $||P^k_-|| = \nu - q_+(\nu) \geq q'_-(\nu)$. Since $P_{ik} = 1$, $W(k) = (i, q(\cdot))$, and $||P^k_+|| = q_+(\nu)$, then $f_k(P) = 1$. On the other hand, since $P_{i'k} = -1$, $W'(k) = (i', q'(\cdot))$, and $||P^k_-|| = \nu - q_+(\nu) \geq q'_-(\nu)$, then $f_k(P) = -1$, contradicting $f_k(P) = 1$. ∎

Lemma 3. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W'_1(k)$ and $W_2(k) = W'_2(k) = q(\cdot)$. Then for each $\nu \in \{1, \ldots, n\}$,* (i) *$q_+(\nu) \leq \nu$, $q_-(\nu) \leq \nu$, $q_+(\nu) + q_-(\nu) = \nu + 1$, and when $\nu \leq n-1$, $q_0(\nu) = q_+(\nu)$, or* (ii) *$(q_+(\nu), q_-(\nu)) \in \{(\nu+1, 1), (1, \nu+1)\}$ and when $\nu \leq n-1$, $(q_+(\nu), q_0(\nu), q_-(\nu)) \in \{(\nu+1, \nu+1, 1), (1, 0, \nu+1)\}$.*

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W'_1(k)$. The proof is in the following four steps.

Step 1. For each $\nu \in \{2, \ldots, n\}$, if $q_+(\nu) \leq \nu$ and $q_-(\nu) \leq \nu$, then $q_+(\nu) + q_-(\nu) = \nu + 1$ and when $\nu \leq n-1$, $q_0(\nu) = q_+(\nu)$.

By Claim 7, $q_+(\nu) + q_-(\nu) \geq \nu + 1$. In order to show $q_+(\nu) + q_-(\nu) = \nu + 1$, suppose $q_+(\nu) + q_-(\nu) \geq \nu + 2$. Let $P \in \mathcal{P}_{\text{Tri}}$ be such that $P_{ik} = 1$, $P_{i'k} = -1$, $||P^k_{+,-}|| = \nu$, and $||P^k_+|| = \nu - q_-(\nu) + 1$ (since $q_+(\nu), q_-(\nu) \leq \nu$ and $q_+(\nu) + q_-(\nu) \geq \nu + 2$, then $q_+(\nu), q_-(\nu) \geq 2$; thus $||P^k_+|| = \nu - q_-(\nu) + 1 = q_+(\nu) - 1 \geq 1$ and similarly $||P^k_-|| = q_-(\nu) - 1 \geq 1$; also note $||P^k_{+,-}|| = \nu \geq 2$; all these guarantee existence of such P). Then $||P^k_+|| = \nu - q_-(\nu) + 1 = q_+(\nu) - 1 < q_+(\nu)$ and $||P^k_-|| = q_-(\nu) - 1 < q_-(\nu)$. Since $P_{ik} = 1$, $W(k) = (i, q(\cdot))$, and $||P^k_+|| < q_+(\nu)$, then $f_k(P) = -1$. Since $P_{i'k} = -1$, $W'(k) = (i', q(\cdot))$, and $||P^k_-|| = \nu - ||P^k_+|| = q_-(\nu) - 1 < q_-(\nu)$, then $f_k(P) = 1$, contradicting $f_k(P) = -1$.

We now show that when $\nu \leq n-1$, $q_0(\nu) = q_+(\nu)$. By (ii) of Claim 3 and the assumption $q_-(\nu) \leq \nu$, $q_0(\nu) \geq 1$. Thus the equation follows directly from (i) of Claim 3.

Step 2. For each $\nu \in \{2, \ldots, n\}$, (i) *if* $q_+(\nu) = \nu + 1$, $q_-(\nu) = 1$; (ii) *if* $q_-(\nu) = \nu + 1$, $q_+(\nu) = 1$.

Suppose $q_+(\nu) = \nu+1$. Since $\nu \geq 2$, there is P such that $||P^k_{+,-}|| = \nu$, $P_{ik} = 1$, $P_{i',k} = -1$, and $||P^k_-|| = 1$ (so $||P^k_+|| = \nu - 1$). Then by i's power $W(k)$, $f_k(P) = -1$. By i''s power $W'(k)$, $q_-(\nu) = 1$. The same argument applies to show the second part.

Step 3. For each $\nu \in \{1, \ldots n-1\}$, (i) $q_0(\nu) = \nu + 1$ *if and only if* $q_+(\nu) = \nu + 1$ *and* $q_-(\nu) = 1$; (ii) $q_0(\nu) = 0$ *if and only if* $q_+(\nu) = 1$ *and* $q_-(\nu) = \nu + 1$.

Part (ii) follows from Claim 3. To prove part (i), suppose $q_0(\nu) = \nu + 1$. Consider P and P' such that $||P^k_{+,-}|| = ||P'^k_{+,-}|| = \nu$, $P_{ik} = P'_{ik} = 0$, $P_{i'k} = 1$, $P'_{i'k} = -1$, $||P^k_+|| = \nu$, and $||P'^k_-|| = 1$. By i's power, $f_k(P) = f_k(P') = -1$. Since $f_k(P) = -1$, by i''s power, $q_+(\nu) > \nu$ and so $q_+(\nu) = \nu + 1$. Also since $f_k(P') = -1$, by i''s power, $q_-(\nu) \leq 1$ and so $q_-(\nu) = 1$. The converse is proven using the same argument in the reverse direction.

Step 4. If $q_0(1) = 0$, *then* $q_+(1) = 1$ *and* $q_-(1) = 2$; *if* $q_0(1) = 1$, *then* $q_+(1) = 1$ *and* $q_-(1) = 1$; *if* $q_0(1) = 2$, *then* $q_+(1) = 2$ *and* $q_-(1) = 1$. *Thus* $(q_+(1), q_0(1), q_-(1)) \in \{(1, 0, 2), (1, 1, 1), (2, 2, 1)\}$.

The two cases for $q_0(1) = 0$ or 2 are shown in Step 3. The remaining case with $q_0(1) = 1$ follows from (i) of Claim 3 and Claim 6. ∎

Remark 2. Lemmas 1 and 3 show that the power on an issue can be either exclusive or non-exclusive. That is, either only one person has the power or everyone has the power. There is no power shared by more than one but not all persons.

Proof of Proposition 1. The characterization of non-exclusive powers in Proposition 1 follows from Lemmas 1 and 3. ∎

Proof of Proposition 2. Uniqueness of systems of powers in Proposition 2 follows from Lemmas 2 and 3, and Proposition 1. ∎

A.2 Proofs of Propositions 3 and 4

Lemma 4. *A rule f is represented by a system of powers $W(\cdot)$ satisfying ladder property if and only if it is represented by an extended system of powers ${}_eW(\cdot)$ such that for each issue*

$k \in M$, the three index sets in ${}_eW_2(k) \equiv (\mathcal{I}_+, \mathcal{I}_0, \mathcal{I}_-)$ are comprehensive and

$$\begin{array}{l} \text{(i)}\ \ (n_1, n_2) \in \mathcal{I}_0 \Rightarrow (n_1 + 1, n_2) \in \mathcal{I}_+; \\ \text{(ii)}\ \ (n_1, n_2) \notin \mathcal{I}_- \Rightarrow (n_2, n_1 - 1) \in \mathcal{I}_0; \\ \text{(iii)}\ \ (n_1, n_2) \notin \mathcal{I}_- \Rightarrow (n_2 + 1, n_1 - 1) \in \mathcal{I}_+. \end{array} \tag{10}$$

Proof. Suppose that person $i \in N$ has the power on the k^{th} issue associated with a consent-quotas function $q(\cdot)$. Then we can construct three comprehensive index sets, $\mathcal{I}_+$, $\mathcal{I}_0$, and $\mathcal{I}_-$ as follows. For each $s \in \{+, 0, -\}$, let $\mathcal{I}_s \equiv \{(n_1, n_2) \in \mathcal{I}^* : n_1 \geq q_s(n_1 + n_2)\}$. Then it is easy to show that (6) implies (2), comprehensiveness of $\mathcal{I}_s$ implies (4) of component ladder property and (10) implies (5) of intercomponent ladder property.

To explain the reverse construction, let $\mathcal{I}_+$, $\mathcal{I}_0$, and $\mathcal{I}_-$ be the three comprehensive sets satisfying (6) and (10). For each $\nu \in \{1, \ldots, n\}$ and each $s \in \{+, 0, -\}$, let

$$q_s(\nu) \equiv \begin{cases} \min\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}_s\}, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}_s\} \neq \emptyset; \\ \nu + 1, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}_s\} = \emptyset. \end{cases}$$

Then this consent-quotas function satisfies the two ladder properties because of comprehensiveness of $\mathcal{I}_+$, $\mathcal{I}_0$, and $\mathcal{I}_-$ and (10). And (6) follows from (2).[18] ∎

Lemma 5. *A rule f represented by an extended system of powers ${}_eW(\cdot)$ satisfies monotonicity if and only if ${}_eW(\cdot)$ satisfies the comprehensiveness property and* (10) *stated in Lemma 4.*

Proof. Let f be a rule represented by an extended system of powers ${}_eW$. Then clearly f satisfies *independence* and so by Proposition 5, f is represented by a profile of decisive structures $(\mathfrak{C}_k)_{k \in M}$. Assume that f satisfies *monotonicity*. Then all decisive structures in $(\mathfrak{C}_k)_{k \in M}$ are monotonic. Let $k \in K$, $i \equiv {}_eW_1(k)$ and $(\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k) \equiv {}_eW_2(k)$. Then by (6), $\mathcal{I}_+^k = \{(|C_1|, |C_2|) \colon (C_1, C_2) \in \mathfrak{C}_k \text{ and } i \in C_1\}$, $\mathcal{I}_0^k = \{(|C_1|, |C_2|) \colon (C_1, C_2) \in \mathfrak{C}_k \text{ and } i \notin C_1 \cup C_2\}$, and $\mathcal{I}_-^k = \{(|C_2|, |C_1|) \colon (C_1, C_2) \notin \mathfrak{C}_k \text{ and } i \in C_2\}$. Comprehensiveness of the three index sets $\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k$ is a direct consequence of monotonicity of the decisive structure $\mathfrak{C}_k$. To show part (i) of (10), let $(n_1, n_2) \in \mathcal{I}_0^k$. Suppose $(n_1 + 1, n_2) \notin \mathcal{I}_+^k$. Let $P \in \mathcal{P}_{\text{Tri}}$ be such that $P_{ik} = 0$, $||P_+^k|| = n_1$, and $||P_-^k|| = n_2$. Then $f_k(P) = 1$. Let $P' \in \mathcal{P}_{\text{Tri}}$ have the same components as P except $P'_{ik} \equiv 1$. Then $P' \geq P$, $||P_+'^k|| = n_1 + 1$, and $||P_-^k|| = n_2$. Since $(n_1 + 1, n_2) \notin \mathcal{I}_+^k$, $f_k(P') = -1$, contradicting *monotonicity* of f.

To show part (ii) of (10), suppose that $(n_1, n_2) \notin \mathcal{I}_-^k$ and $(n_2, n_1 - 1) \notin \mathcal{I}_0^k$. Let $P \in \mathcal{P}_{\text{Tri}}$ be such that $P_{ik} = -1$, $||P_-^k|| = n_1$, and $||P_+^k|| = n_2$. Then $f_k(P) = 1$. Let $P' \in \mathcal{P}_{\text{Tri}}$ have the same components as P except $P'_{ik} \equiv 0$. Then $P' \geq P$, $||P_+'^k|| = n_2$, and $||P_-^k|| = n_1 - 1$. Since $(n_2, n_1 - 1) \notin \mathcal{I}_0^k$, $f_k(P') = -1$, contradicting *monotonicity* of f.

To show (iii) of (10), suppose that $(n_1, n_2) \notin \mathcal{I}_-^k$ and $(n_2 + 1, n_1 - 1) \notin \mathcal{I}_+^k$. Let $P \in \mathcal{P}_{\text{Tri}}$ be such that $P_{ik} = -1$, $||P_-^k|| = n_1$, and $||P_+^k|| = n_2$. Then $f_k(P) = 1$. Let $P' \in \mathcal{P}_{\text{Tri}}$ have the same components as P except $P'_{ik} \equiv 1$. Then $P' \geq P$, $||P_+'^k|| = n_2 + 1$, and $||P_-^k|| = n_1 - 1$. Since $(n_2 + 1, n_1 - 1) \notin \mathcal{I}_0^k$, $f_k(P') = -1$, contradicting *monotonicity* of f.

To prove the converse, assume that ${}_eW$ satisfies the comprehensiveness property and (10) stated in Lemma 4. In order to prove *monotonicity* of f, let $P' \geq P$ and $k \in M$ be such that

[18] The proof is available upon request.

$f_k(P) = 1$. We only have to show $f_k(P') = 1$. Let $i \equiv {}_eW(k)$ and $(\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k) \equiv {}_eW_2(k)$. When $P'_{ik} = P_{ik}$, it follows directly from the comprehensiveness condition of the three sets $\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k$ that $f_k(P') = 1$. There are two remaining cases.

Case 1. $P_{ik} = 0 \neq P'_{ik}$ *and* $(||P_+^k||, ||P_-^k||) \in \mathcal{I}_0^k$. Then $P'_{ik} = 1$. Hence $||P'^k_+|| \geq ||P_+^k|| + 1$ and $||P'^k_-|| \leq ||P_-^k||$. By comprehensiveness of $\mathcal{I}_+^k$ and part (i) of (10), $(||P'^k_+||, ||P'^k_-||) \in \mathcal{I}_+^k$. Therefore $f_k(P') = 1$.

Case 2. $P_{ik} = -1 \neq P'_{ik}$ *and* $(||P_-^k||, ||P_+^k||) \notin \mathcal{I}_-^k$. Then either $P'_{ik} = 0$ or $P'_{ik} = 1$. If $P'_{ik} = 0$, $||P'^k_+|| \geq ||P_+^k||$ and $||P'^k_-|| \leq ||P_-^k|| - 1$. Then by comprehensiveness of $\mathcal{I}_-^k$ and part (ii) of (10), $(||P'^k_+||, ||P'^k_-||) \in \mathcal{I}_0^k$. Thus, $f_k(P') = 1$. If $P'_{ik} = 1$, $||P'^k_+|| \geq ||P_+^k|| + 1$ and $||P'^k_-|| \leq ||P_-^k|| - 1$. Then by comprehensiveness of $\mathcal{I}_-^k$ and part (iii) of (10), $(||P'^k_+||, ||P'^k_-||) \in \mathcal{I}_+^k$. Therefore $f_k(P') = 1$. ∎

Proof of Proposition 3. Proposition 3 follows directly from Lemmas 4 and 5. ∎

Proof of Proposition 4. Consider a rule f represented by a system of powers $W \in \mathfrak{W}^\Lambda$. Let $\lambda(\cdot) \equiv W_1(\cdot)$. Let $\pi\colon N \to N$ be a permutation on N and $\delta\colon M \to M$ a permutation on M such that for each $i \in N$, δ maps $\lambda^{-1}(i)$ onto $\lambda^{-1}(\pi(i))$. Then because of the ontoness property of δ, $i \in N$ and $\pi(i)$ are associated with the same number of issues under λ. Thus by horizontal equality, for each $k \in \lambda^{-1}(i)$, i's power on the k^{th} issue and $\pi(i)$'s power on the $\delta(k)^{\text{th}}$ issue are associated with the same consent-quotas function, that is, $W_2(k) = W_2(\delta(k))$. Denote the common consent-quotas function by $q(\cdot)$. For each $P \in \mathcal{P}_{\text{Tri}}$, $||P_+^{\delta(k)}|| = ||{}^\delta_\pi P_+^k||$ and $||P_-^{\delta(k)}|| = ||{}^\delta_\pi P_-^k||$. Thus, $q(||P_{+,-}^{\delta(k)}||) = q(||{}^\delta_\pi P_+^k||)$ and ${}^\delta_\pi P_{ik} = P_{\pi(i)\delta(k)}$. Therefore, $f_k\left({}^\delta_\pi P\right) = f_{\delta(k)}(P)$. This shows that f satisfies *symmetric linkage* associated with λ. The converse can be proven similarly. ∎

A.3 Proofs of Proposition 7 and Theorem 1

Proof of Proposition 7. Using the same argument as in the proof of Proposition 4, we can show that a rule represented by an extended system of powers in ${}_e\mathfrak{W}^\Lambda$ satisfies *symmetric linkage* if and only if the extended system satisfies horizontal equality. Clearly, any rule represented by an extended system of powers satisfies *independence*.

To prove the converse, consider a rule f satisfying *independence* and *symmetric linkage*. Then by Proposition 5, f is represented by a profile of decisive structures $(\mathfrak{C}_k)_{k\in M}$. Let f satisfy *symmetric linkage* with respect to $\lambda \in \Lambda$. We identify an extended system of powers of f and complete the proof in two steps.

Step 1. For each pair $i, j \in N$ with $|\lambda^{-1}(i)| = |\lambda^{-1}(j)|$, each $k \in \lambda^{-1}(i)$, each $l \in \lambda^{-1}(j)$, and each $(C_1, C_2), (C'_1, C'_2) \in \mathfrak{C}^*$ with $|C_1 \cap \{i\}| = |C'_1 \cap \{j\}|$ and $|C_2 \cap \{i\}| = |C'_2 \cap \{j\}|$ (or equivalently, $[i \in C_1 \Leftrightarrow j \in C'_1]$ and $[i \in C_2 \Leftrightarrow j \in C'_2]$), if $|C_1| = |C'_1|$ and $|C_2| = |C'_2|$, then $(C_1, C_2) \in \mathfrak{C}_k \Leftrightarrow (C'_1, C'_2) \in \mathfrak{C}_l$.

Let $i, j \in N$, $k \in \lambda^{-1}(i)$, $l \in \lambda^{-1}(j)$, and $(C_1, C_2), (C'_1, C'_2) \in \mathfrak{C}^*$ be given as above. Consider the case $i \in C_1$ and $j \in C'_1$ (the proofs for the other cases are similar). Suppose $(C_1, C_2) \in \mathfrak{C}_k$. Let P be such that $N(P_+^k) \equiv C_1$ and $N(P_-^k) \equiv C_2$. So $f_k(P) = 1$. Since $|C_1| = |C'_1|$ and $|C_2| = |C'_2|$, there is a permutation π on N such that $\pi(i) = j$, $\pi(j) = i$, $\pi(C_1) = C'_1$, and $\pi(C_2) = C'_2$. Since $|\lambda^{-1}(i)| = |\lambda^{-1}(j)|$, there is a permutation δ on

M such that $\delta\left(\lambda^{-1}(j)\right) = \lambda^{-1}(i)$, $\delta(\lambda^{-1}(i)) = \lambda^{-1}(j)$, $\delta(l) = k$, and for all other $k' \in M \backslash [\lambda^{-1}(i) \cup \lambda^{-1}(j)]$, $\delta(k') = k'$. Then $N({}^{\delta}_{\pi}P^{l}_{+}) = \pi^{-1}(N(P^{\delta(l)}_{+})) = \pi^{-1}(C_1) = C'_1$. Similarly, $N({}^{\delta}_{\pi}P^{l}_{-}) = C'_2$. By λ-*symmetry*, $f_l\left({}^{\delta}_{\pi}P\right) = f_{\delta(l)}(P) = f_k(P) = 1$. Therefore, $(C'_1, C'_2) \in \mathfrak{C}_l$. The proof of the opposite direction is similar.

By Step 1, for each $i \in N$ and each pair $k, l \in \lambda^{-1}(i)$, $\mathfrak{C}_k = \mathfrak{C}_l$.

Step 2. Rule f is represented by an extended system of powers in ${}_{e}\mathfrak{W}^{\Lambda}$ satisfying horizontal equality.

Let N/λ be the partition of N such that for each pair $i, j \in N$, i and j are in the same set $G \in N/\lambda$ if and only if $|\lambda^{-1}(i)| = |\lambda^{-1}(j)|$. For each $G \in N/\lambda$, let $K_G \equiv \{k \in M : \lambda(k) \in G\}$ be the set of issues linked to a person in G under λ. Then $M/\lambda \equiv \{K_G : G \in N/\lambda\}$ is a partition of M. For each $K \in M/\lambda$, pick $k \in K$ and let $i \equiv \lambda(k)$. Let $\mathcal{I}^K_+ \equiv \{(|C_1|, |C_2|)\colon (C_1, C_2) \in \mathfrak{C}_k \text{ and } i \in C_1\}$, $\mathcal{I}^K_0 \equiv \{(|C_1|, |C_2|)\colon (C_1, C_2) \in \mathfrak{C}_k \text{ and } i \notin C_1 \cup C_2\}$, and $\mathcal{I}^K_- \equiv \{(|C_2|, |C_1|)\colon (C_1, C_2) \notin \mathfrak{C}_k \text{ and } i \in C_2\}$. For each $l \in K \in M/\lambda$, let ${}_{e}W_2(l) \equiv (\mathcal{I}^K_+, \mathcal{I}^K_0, \mathcal{I}^K_-)$. Let ${}_{e}W_1(\cdot) \equiv \lambda$ and ${}_{e}W(\cdot) \equiv ({}_{e}W_1(\cdot), {}_{e}W_2(\cdot))$. Then by construction, ${}_{e}W(\cdot)$ satisfies horizontal equality. We next show that for each $P \in \mathcal{P}_{\text{Tri}}$, each $K \in M/\lambda$, and each $l \in K$, if $\lambda(l) = j \in N$,

$$\begin{array}{l} \text{(i) when } P_{jl} = 1,\ f_l(P) = 1 \Leftrightarrow (||P^l_+||, ||P^l_-||) \in \mathcal{I}^K_+; \\ \text{(ii) when } P_{jl} = 0,\ f_l(P) = 1 \Leftrightarrow (||P^l_+||, ||P^l_-||) \in \mathcal{I}^K_0; \\ \text{(iii) when } P_{jl} = -1,\ f_l(P) = -1 \Leftrightarrow (||P^l_-||, ||P^l_+||) \in \mathcal{I}^K_-. \end{array} \tag{11}$$

When $j = i$, Step 1 says that the decision on the k^{th} issue relies on person i's opinion, the number of agreeing persons, and the number of disagreeing persons. Therefore, since for each $l \in \lambda^{-1}(i)$, $\mathfrak{C}_l = \mathfrak{C}_k$, then (11) holds when $j = i$. When $j \in G \backslash \{i\}$, Step 1 says that for each $l \in \lambda^{-1}(j)$, the decision on the l^{th} issue is made symmetrically to the decision on the k^{th} issue. Therefore, (11) holds also for j and l. ∎

Proof of Theorem 1. Theorem 1 follows directly from Propositions 3, 4 and 7, and Lemmas 4 and 5. ∎

A.4 Proofs of Theorems 5 and 6

Proof of Theorem 5. Let f be a rule on $\mathcal{P}_{\text{Tri}}$ (or $\mathcal{R}_{\text{Tri}}$, recall that we will treat each opinion matrix as a profile of trichotomous preference relations) satisfying the three axioms (the proof for $\mathcal{P}_{\text{Di}}$ or $\mathcal{R}_{\text{Di}}$ is essentially the same). Then by Proposition 7, f is represented by an extended system of powers ${}_{e}W(\cdot)$ and ${}_{e}W_1(\cdot) \in \Lambda$. Let $\lambda(\cdot) \equiv {}_{e}W_1(\cdot)$. Without loss of generality, we assume $N \subseteq M$ (since the number of objects linked to a person is constant across persons, we may label at least one object by the label of the person linked to it) and for each $i \in \{1, \ldots, n\}$, $\lambda(i) = i$. By Proposition 7 and the assumption on λ, there exist three index sets $\mathcal{I}_+$, $\mathcal{I}_0$, and $\mathcal{I}_-$ such that for each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$, if $i \equiv \lambda(k)$,

$$\begin{array}{l} \text{(i) when } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow \left(||P^k_+||, ||P^k_-||\right) \in \mathcal{I}_+; \\ \text{(ii) when } P_{ik} = 0,\ f_k(P) = 1 \Leftrightarrow \left(||P^k_+||, ||P^k_-||\right) \in \mathcal{I}_0; \\ \text{(iii) when } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow \left(||P^k_-||, ||P^k_+||\right) \in \mathcal{I}_-. \end{array} \tag{12}$$

$$P \equiv \begin{pmatrix} 1 & 0 & -1 & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}; \; {}^{\delta}_{\pi}P = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 1: Construction of P in the proof of Theorem 5. An example with $|N| = |M| = 6$, $t_1 = 3$, $t_2 = 2$, $i = 1$, and $j = 2$. Let $\pi\colon N \to N$ be the transposition of 1 and 2 and $\delta\colon M \to M$ the same transposition.

Claim 1. For each $s \in \{+, 0, -\}$,

$$\begin{aligned} &\{(t_1, t_2) \in \mathcal{I}^* : t_1 > t_2\} \subseteq \mathcal{I}_s; \\ &\{(t_1, t_2) \in \mathcal{I}^* : t_1 < t_2\} \cap \mathcal{I}_s = \emptyset. \end{aligned} \tag{13}$$

Proof. Let $(t_1, t_2) \in \mathcal{I}^*$ be such that $t_1 > t_2$. Suppose by contradiction $(t_1, t_2) \notin \mathcal{I}_+$. Let $[0] \equiv n$. For each $l \in \{1, \ldots, n\}$, let $[l] \equiv l$, $[n + l] \equiv l$, and $[-l] \equiv [n - l]$. Let P be the opinion matrix such that for each $i \in \{1, \ldots, n\}$, if $l \in \{0, 1, \ldots, t_1 - 1\}$, $P_{[i+l]i} = 1$; if $l = t_1, \ldots, t_1 + t_2 - 1$, $P_{[i+l]i} = -1$; if $l = t_1 + t_2, \ldots, n$, $P_{[i+l]i} = 0$; and for each $k \in M\backslash\{1, \ldots, n\}$ and each $i \in N$, $P_{ik} = -1$. See Figure 1 for an illustration of P. Then for each $i \in \{1, \ldots, n\}$, there are t_1 persons, $\{[i], [i + 1], \ldots, [i + t_1 - 1]\}$, who have the positive opinion on the i^{th} issue, t_2 persons, $\{[i + t_1], \ldots, [i + t_1 + t_2 - 1]\}$, who have the negative opinion, and $n - t_1 - t_2$ remaining persons with the null opinion. Hence for each $i \in \{1, \ldots, n\}$, $||P^i_+|| = t_1$ and $||P^i_-|| = t_2$. Let $i, j \in \{1, \ldots, n\}$. Let $\pi\colon N \to N$ and $\delta\colon M \to M$ be two permutations on N and on M transposing i and j. Then the i^{th} and the j^{th} columns in ${}^{\delta}_{\pi}P$ are obtained by making an one-to-one and onto switch between the i^{th} and the j^{th} columns in P, not necessarily preserving the row positions of entries.[19] Thus, $||{}^{\delta}_{\pi}P^i_+|| = ||P^j_+||$, $||{}^{\delta}_{\pi}P^i_-|| = ||P^j_+||$, $||{}^{\delta}_{\pi}P^j_+|| = ||P^i_+||$, and $||{}^{\delta}_{\pi}P^j_-|| = ||P^i_+||$. By *symmetry*, $f_i\left({}^{\delta}_{\pi}P\right) = f_j(P)$ and $f_j\left({}^{\delta}_{\pi}P\right) = f_i(P)$. Since $||P^i_+|| = ||P^j_+||$ and $||P^i_-|| = ||P^j_-||$, then $||P^i_+|| = ||{}^{\delta}_{\pi}P^j_+||$, $||P^i_-|| = ||{}^{\delta}_{\pi}P^i_-||$, $||P^j_+|| = ||{}^{\delta}_{\pi}P^j_+||$, and $||P^j_-|| = ||{}^{\delta}_{\pi}P^j_-||$. So $f_i(P) = f_i\left({}^{\delta}_{\pi}P\right)$ and $f_j(P) = f_j\left({}^{\delta}_{\pi}P\right)$. Hence $f_i(P) = f_j(P)$. Since $(t_1, t_2) \notin \mathcal{I}$, $f_N(P) = (-1, \ldots, -1)$. On the other hand, by *Pareto efficiency*, $f_{M\backslash N} = (-1, \ldots, -1)$. For each $i \in N$, let $U_i(\cdot)$ be the representation of the trichotomous preference relation P_i. Then for each $i \in N$, $U_i(f(P)) = 0$. Let x be such that $x_N \equiv (1, \ldots, 1)$ and $x_{M\backslash N} \equiv (-1, \ldots, -1)$. Then for each $i \in N$, $U_i(x) = t_1 - t_2 > 0$, contradicting *Pareto efficiency*.

Let $(t_1, t_2) \in \mathcal{I}^*$ be such that $t_1 < t_2$. Suppose by contradiction $(t_1, t_2) \in \mathcal{I}_+$. Then using the same argument as above, we show $f_N(P) = (1, \ldots, 1)$ and $f_{M\backslash N}(P) = (-1, \ldots, -1)$. Let $x \equiv (-1, \ldots, -1)$. Then for each $i \in N$, $U_i(f(P)) = t_1 - t_2 < 0 = U_i(x)$, contradicting *Pareto efficiency*.

[19]Note that P_{ii} and P_{ji} in the i^{th} column are switched into P_{jj} and P_{ij} in the j^{th} column respectively. Other entries in the i^{th} column are switched into the entries in the j^{th} column in the same rows.

Similar arguments can be used to prove the same properties for $\mathcal{I}_0$ and $\mathcal{I}_-$. ♦

Note that the properties stated in (13) imply comprehensiveness of the three index sets. Finally, for each $s \in \{+, 0, -\}$, let $q_s(\nu) \equiv \min\{t_1 : (t_1, \nu - t_1) \in \mathcal{I}_s\}$ for each ν. Then (13) implies (7) and (8). Because of comprehensiveness of the three index sets, (12) implies (2). ∎

Proof of Theorem 6. Let f be a rule over $\mathcal{P}_{\text{Tri}}$ satisfying the four axioms (the proof for $\mathcal{P}_{\text{Di}}$ or $\mathcal{R}_{\text{Di}}$ is essentially the same). By Proposition 7, f is represented by an extended system of powers ${}_eW(\cdot) \in {}_e\mathfrak{W}^\Lambda$. Then by *neutrality*, for each pair $l, k \in M$, ${}_eW_2(l) = {}_eW_2(k)$. Thus there exist three index sets $\mathcal{I}_+$, $\mathcal{I}_0$, and $\mathcal{I}_-$ such that for each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$, if (12) holds for $i \equiv \lambda(k)$. Using essentially the same argument as in the proof of Theorem 5, we can show that f is represented by a quasi-plurality system of powers. Because of *neutrality*, the system is either non-exclusive or monocentric. ∎

References

[1] Barberà, S., H. Sonnenschein, and L. Zhou (1991), "Voting by committees", *Econometrica* 59: 595-609

[2] Blau, J.H. (1975), "Liberal values and independence", *Review of Economic Studies* 42: 395-402

[3] Deb, R., P.K. Pattanaik, and L. Razzolini (1997), "Game forms, rights, and the efficiency of social outcomes", *Journal of Economic Theory* 72: 74-95

[4] Gaertner, W., P.K. Pattanaik, and K. Suzumura (1992), "Individual rights revisited", *Economica* 59 (234): 161-177

[5] Gibbard, A. (1974), "A Pareto-consistent libertarian claim", *Journal of Economic Theory* 7: 388-410

[6] Ju, B.-G. (2003), "A characterization strategy-proof voting rules for separable weak orderings", *Social Choice and Welfare* 21(3): 469-499

[7] Ju, B.-G. (2005), "An Efficiency Characterization of Plurality Social Choice on Simple Preference Domains", *Economic Theory* 26(1): 115-128

[8] Kasher, A. and A. Rubinstein (1997), "On the question 'Who is a j', a social choice approach", *Logique et Analyse* 160:385-395

[9] Rubinstein, A. and P.C. Fishburn (1986), "Algebraic aggregation theory", *Journal of Economic Theory* 38: 63-77

[10] Samet, D. and D. Schmeidler (2003), "Between liberalism and democracy", *Journal of Economic Theory* 110(2):213-233

[11] Sen, A.K. (1970), "The impossibility of a Paretian Liberal", *Journal of Political Economy* 78: 152-157

[12] Sen, A.K. (1976), "Liberty, unanimity and rights", *Economica* 43: 217-245

[13] Sen, A.K. (1983), "Liberty and social choice", *Journal of Philosophy* 80(1):5-28

[14] Thomson, W. (2001), "On the axiomatic method and its recent applications to game theory and resource allocation", *Social Choice and Welfare* 18:327-386