Lower bounds and recursive methods for the problem of adjudicating conflicting claims*

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Abstract

For the problem of adjudicating conflicting claims, we study lower bounds on the awards of each agent. We introduce some lower bounds and properties to test their behavior, and note which lower bounds satisfy each property. Then, we consider extending a lower bound in the following way: (i) for each problem compute the lower bound and revise the problem accordingly; (ii) compute the bound of the revised problem; (iii) the extension assigns the sum of the computed bounds. A "recursive extension" is obtained by recursive application of this operation. We provide necessary and sufficient conditions, on a lower bound, for there to be a unique rule satisfying its recursive extension. We show that, under these conditions, the rule satisfying the recursive extension of a bound is the unique rule satisfying the following property: the awards recommended by a rule should be obtainable in two ways: (i) directly applying the rule to the problem or (ii) first assigning the bound and revising the problem accordingly, and then applying the rule to the revised problem. Then we study which properties, imposed on a lower bound, are necessarily satisfied by its recursive extension. Finally we turn to two specific lower bounds and find that, for two-agent problems, their recursive extensions single out two well-know rules.

Keywords: Claims problems, Lower bounds, Recursive methods.

JEL Classification Numbers: C79, D63, D74

1 Introduction

When a group of agents have legitimate claims over some resource and there is not enough of the resource to fulfill all of the claims, how should the resource be divided among the agents? When the resource consist of a positive amount of a single infinitely divisible good we have a "claims problem". A "rule" recommends, for each problem, a division of the resource among the agents. A "lower bound" recommends, for each problem and to each agent, a minimal amount of the resource that she should receive. A rule satisfies a lower bound if, for each problem and each agent, the rule assigns her at least as much as the lower bound does. Several lower bounds have been studied in the literature. Rules satisfying them (along with other properties) have been proposed.

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In recent work Dominguez and Thomson (2006) study a specific lower bound introduced by Moreno-Ternero and Villar (2004). This lower bound assigns, to each agent, the minimum between (i) her claim divided by the number of agents and (ii) the amount to divide divided by the number of agents. They show that, even though many rules satisfy the lower bound, only one satisfies the following invariance property: the awards recommended by the rule should be obtainable in two ways: (i) applying the rule directly or (ii) first assigning the lower bound, then applying the rule an appropiately revised problem.

In this paper we study the invariance property for lower bounds in general. Lower bounds are used to protect people from being treated too unfairly. If a lower bound assigns a positive amount to an agent, then a rule satisfying the bound assigns her a positive amount. It turns out that a sufficient condition on a lower bound, under which the invariance property yields a unique rule, is that the bound assigns, for each problem and to each agent, a positive amount. Moreover, the weaker condition that the bound assigns, for each problem to at least one agent, a positive amount is not only sufficient but also necessary for uniqueness. Our main theorem formalizes this claim. It also provides a new interpretation of the invariance property: if after assigning to each agent the amount assigned to her by the lower bound and revising the problem accordingly, the lower bound of the revised problem assigns positive amounts, then the lower bound is too small and we should extend it. The "recursive extension of a bound" is obtained by repeated extensions of the bound. Whenever the recursive extension of a bound singles out a rule, it is the unique rule satisfying the invariance property with respect to the bound.

We also study properties of good behavior for lower bounds. Most of the properties in the literature are formulated for rules, usually, they can be directly applied to lower bounds and their motivation remains valid. Some properties when imposed on a bound are automatically satisfied by its recursive extension, and we say that they are "inherited". We undertake a systematic investigation of inheritance. Some properties are inherited on their own, while some are inherited only when imposed together with other properties.

Finally, we focus on two specific lower bounds. The first assigns to each agent $1/n$ of the minimum between the smallest claim and the amount to divide, where n is the number of agents. We call this bound the min lower bound. When there are only two agents, there is a unique rule satisfying the recursive extension of the min lower bound. If there are more than two agents, there are several rules satisfying it, but there is a unique "consistent" rule satisfying it. It is a well known rule in the literature as the *constrained equal awards rule.*

The second lower bound is defined for two-agent situations, it assigns half of the amount to divide to the agent with the largest claim, and half of the difference between the amount to divide and the absolute value of the difference of the claims, whenever this amount is positive and 0 otherwise, to the agent with the smallest claim. There is a unique two-agent rule satisfying its recursive extension. If there are more than two agents there is a unique "consistent" rule satisfying it. It is a well known rule in the literature as the *constrained equal losses rule.*

2 The model

There is a social **endowment** $E \in \mathbb{R}_+$ of an infinitely divisible good.[1] The endowment has to be divided among a set N of **agents**.[2] Each agent $i \in N$ has a **claim** $c_i \in \mathbb{R}_+$

[1]The set $\mathbb{R}_+$ denotes the non-negative reals.

[2]We refer to the set of agents as the population of a problem.

on the endowment. Each agent has strictly monotonic preferences over the amount she receives. The endowment is not sufficient to honor all the claims. A **problem** is a vector $(c, E) \in \mathbb{R}_+^{|N|} \times \mathbb{R}_+$ such that $E \leq \sum_{i \in N} c_i$. Let $\mathcal{C}^N$ be the set of all problems with population N.[3]

Definition 1. A **rule** is a function $\varphi : \mathcal{C}^N \to \mathbb{R}^{|N|}$, which maps each problem $(c, E) \in \mathcal{C}^N$, to an awards vector $\varphi(c, E) \in \mathbb{R}^{|N|}$ such that, for each $(c, E) \in \mathcal{C}^N$,

(i) $0 \leqq \varphi(c, E) \leqq c$,[4]

(ii) $\sum_{i \in N} \varphi_i(c, E) = E$.

The **set of awards vectors** for the problem (c, E), denoted $X(c, E)$, is the set of vectors satisfying conditions (i) and (ii). The inequalities in (i) are the **non-negativity** and **claims boundedness** restrictions. Condition (ii) is **efficiency**. The set of vectors satisfying *efficiency* is denoted $F(c, E)$. For some problems the set of awards vectors consists of a single element.[5] Such problems are **degenerate**, and we obtain the awards vector by the definition of a rule. Whenever the set of awards vectors contains at least 2 elements the problem is **non-degenerate**.

A graphical representation of a rule is by means of its **paths of awards** (see Figure 1). Given a rule φ, for each claims vector c, the path of awards of φ for c is the image of a function $p(\cdot|\varphi, c) : [0, \sum_i c_i] \to X(c, \cdot)$, which maps each endowment E with $0 \leq E \leq \sum_i c_i$, into the awards vector recommended by the rule. It describes the path followed by the awards vectors when the endowment varies from 0 to the sum of the claims.

Definition 2. A **lower bound** is a function $b : \mathcal{C}^N \to \mathbb{R}^{|N|}$, which maps each problem $(c, E) \in \mathcal{C}^N$, to a rights vector $b(c, E) \in \mathbb{R}^{|N|}$ such that, for each $(c, E) \in \mathcal{C}$,

(i) $0 \leqq b(c, E) \leqq c$,

(ii) $\sum_{i \in N} b_i(c, E) \leq E$.

The **set of rights vectors** for the problem (c, E) is the set of vectors satisfying conditions (i) and (ii). Condition (ii) is **budget balancedness**. A lower bound can be interpreted as an *inefficient* rule. Conversely, a rule can be interpreted as an *efficient* lower bound.

Just like the paths of awards of a rule provide a graphical representation for it, a graphical representation for a lower bound is by means of its **paths of acceptable vectors** (see Figure 1). Given a lower bound b, for each claims vector c, the path of acceptable vectors of b for c is the image of a (convex valued) correspondence $p(\cdot|b, c) : [0, \sum_i c_i] \to F(c, \cdot)$, which maps each endowment E with $0 \leq E \leq \sum_i c_i$, into the set of *efficient* vectors dominating the rights vector recommended by the lower bound. It can be described by the set of vectors satisfying $|N|$ restrictions. Each restriction assigns to a distinct agent her right, and allows any division of the remainder among the other agents.

We say that a rule **satisfies** a lower bound if, for each problem, the awards vector recommended by the rule weakly dominates the rights vector recommended by the lower bound. Otherwise it **fails** it. Graphically, there is a simple procedure to determine whether

[3] This model was first introduced by O'Neill (1982).

[4] Vector inequalities: $x \geqq y \Leftrightarrow$ for each $i \in N$, $x_i \geq y_i$; $x \geq y \Leftrightarrow x \geqq y$ and $x \neq y$; $x > y \Leftrightarrow$ for each $i \in N$, $x_i > y_i$.

[5] Such problems arise when $E = 0$ or $E = \sum_{i \in N} c_i$, or when all but one of the claims are equal to 0.

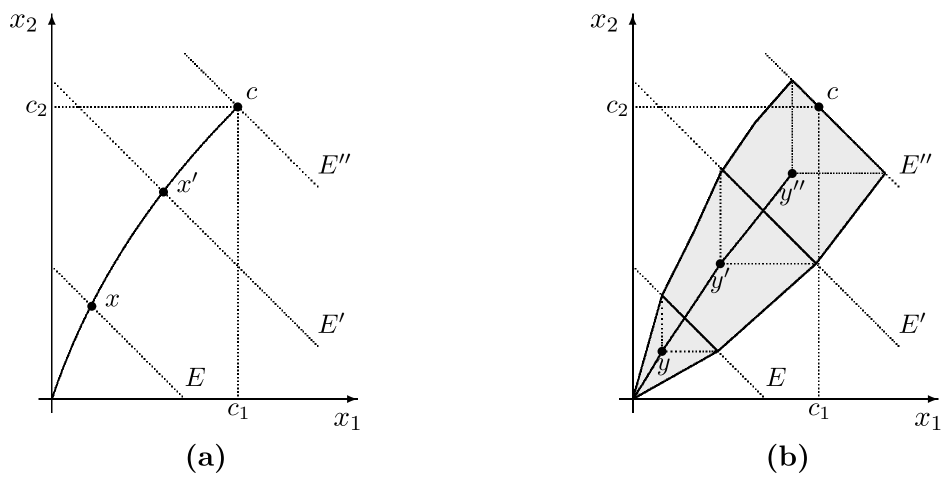

Figure 1: **Paths of awards and paths of acceptable vectors.** (a) Path of awards of φ for c. A single path traced out by the awards vectors assigned by the rule as the endowment varies from 0 to the sum of the claims; the awards vectors $\varphi(c, E)$ and $\varphi(c, E')$ are the vectors x and x' respectively. (b) Path of acceptable vectors of b for c. For $|N| = 2$ each restriction traces a path, the first assigns her right to agent 1 and the remainder to agent 2, the second assigns her right to agent 2 and the remainder to agent 1; the rights vectors $b(c, E)$, $b(c, E')$, and $b(c, E'')$ are the vectors y, y', and y'' respectively. The path connecting the vectors y, y' and y'' is the path traced by the lower bound vectors as the endowment varies from 0 to the sum of the claims.

or not a rule satisfies a lower bound. It is to verify if, for each claims vector, the path of awards of the rule takes values in the path of acceptable vectors of the lower bound (see Figure 1).

In the variable population model, we allow for an infinite set of potential agents $\mathbb{N}$. However, in a given problem, there is a finite set of agents N. Let $\mathcal{N}$ denote the set of all finite subsets of $\mathbb{N}$. A problem is determined by a population $N \in \mathcal{N}$ and a vector $(c, E) \in \mathcal{C}^N$. A rule is now defined on $\mathcal{C} = \bigcup_{N \in \mathcal{N}} \mathcal{C}^N$. Similarly a lower bound is defined on $\mathcal{C}$.

3 Lower bounds and properties

In this section we introduce some existing lower bounds and formulate a new one. Also, we present properties for the lower bounds. The desirability of the properties can be evaluated from different points of view: horizontal equity, monotonicity, and operational properties. The properties are used as tests on how well-behaved the lower bounds are.

Many desirable properties for rules are also desirable for lower bounds. Conversely, many desirable properties for lower bounds are desirable for rules. For instance, in most applications, it is desirable that agents with the same claims be treated equally; *equal treatment of equals* says that a rule assigns the same awards to agents with the same claims; in such situations, it also seems reasonable that a lower bound recommends the same rights to those agents. We introduce some of the existing properties for rules,[6] and (if necessary) modify them to accommodate lower bounds. We also introduce new properties which are desirable for lower bounds. After formulating each property, we note which lower bounds

[6]Most of the literature on claims problems has been axiomatic. For a discussion on the axiomatic method, applied to this problem and other economic problems, see Thomson (2001).

satisfy it. The proofs are omitted since they are straightforward.

3.1 Lower bounds

Before we introduce some lower bounds, note that, the definition of a rule includes the requirement that, for each problem, awards are non-negative. This restriction can be seen as a formal lower bound. It is a basic property that all rules should satisfy, which is why we embedded it into the definition of a rule. In fact, it is such a basic property that we also impose it for lower bounds.

The first lower bound assigns, to each agent, the difference between the endowment and the sum of the claims of the other agents, or zero if this difference is negative. For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, agent i's **minimal right** is $m_i(c, E) = \max\{E - \sum_{j \in N \setminus i} c_j, 0\}$. The **minimal rights lower bound** (Curiel, Maschler and Tijs 1987) is given by:

$$m(c, E) = (m_i(c, E))_{i \in N}.$$

Minimal rights is a very weak bound (see Figure 2). It is an implication of the *non-negativity*, *claim-boundedness*, and *efficiency* requirements. Thus, all rules satisfy *minimal rights.*

Our next bound is mentioned in the introduction. For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, agent i's truncated claim is $t_i(c, E) = \min\{c_i, E\}$. The vector of **truncated claims** is $t(c, E) \equiv (t_i(c, E))_{i \in N}$. The **reasonable lower bound** (Moreno-Ternero and Villar 2004) is given by:

$$r(c, E) = \frac{1}{|N|} t(c, E).$$

Most rules in the literature satisfy *reasonable lower bound.* The *constrained equal awards rule* (which selects $x \in X(c, E)$ such that for some $\lambda \in \mathbb{R}_+$, $x = (\min\{c_i, \lambda\})_{i \in N}$), the *random arrival rule* (which selects the average of the awards vectors obtained by imagining agents arriving one at a time and fully compensating them until the endowment runs out, under the assumption that all orders are equally likely), and the *Talmud rule* (which selects $x \in X(c, E)$ such that $x = (min\{\frac{c_i}{2}, \lambda\})_{i \in N}$ if $E \leq \frac{\sum c_i}{2}$, and $x = \frac{c}{2} + (\max\{\frac{c_i}{2} - \lambda, 0\})_{i \in N}$ otherwise) satisfy *reasonable lower bound*, and others do to.[7] Unfortunately, the *proportional* rule (which selects $x \in X(c, E)$ such that for some $\lambda \in \mathbb{R}_+$, $x = \lambda c$) and the *constrained equal losses* rule (which selects $x \in X(c, E)$ such that for some $\lambda \in \mathbb{R}_+$, $x = (\max\{c_i - \lambda, 0\})_{i \in N}$) fail *reasonable lower bound* (see Figure 2).

The next bound is new to the literature. It is "weaker" than *reasonable lower bound.*[8] The bound is very useful to illustrate our results, moreover it also has some intuitive appeal.

To motivate it, start from a problem $(c, E) \in \mathcal{C}^N$ and consider a reference situation where each agent's claim is equal to the smallest claim in the original problem, and the endowment remains the same.[9] In such situation agents should be treated equally; if the endowment is smaller than the common claim, equal division should prevail; if the endowment is greater

[7] Dominguez and Thomson (2006) provide some general ways of identifying rules satisfying *reasonable lower bound.* They also provide intuition for the bound.

[8] Even though *min lower bound* is weaker than *reasonable lower bound*, the *proportional* rule and the *constrained equal losses* rule violate *min lower bound* (see Figure 2).

[9] The reference situation may not be a problem since the sum of the claims may be smaller than the endowment. Reference to this situation can help to determine the rights in the original problem.

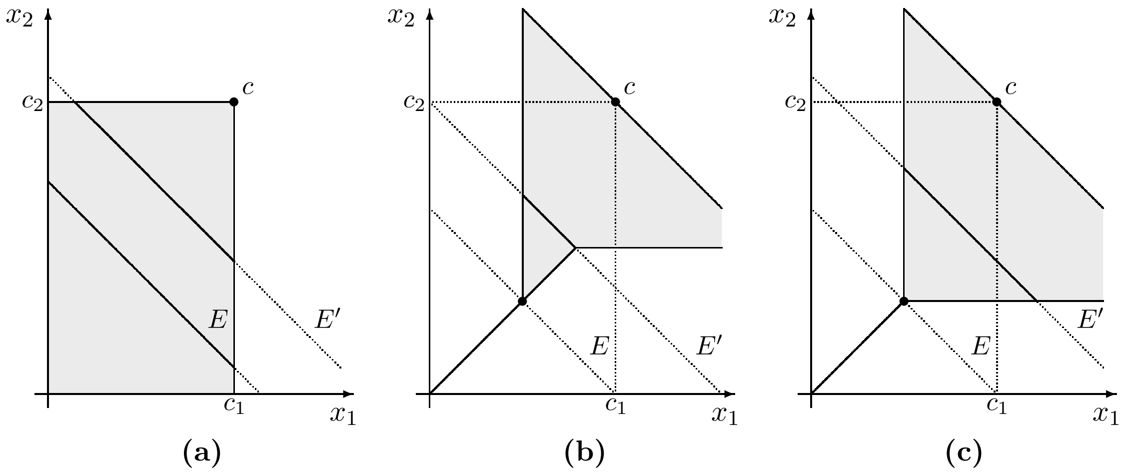

Figure 2: **Graphical representations of some lower bounds for $|\mathbf{N}| = \mathbf{2}$.** For each problem the acceptable vectors satisfy $|N| = 2$ restrictions. Each restriction traces a path. The first assigns her right to agent 1, and the remainder to agent 2. The second is symmetrically defined. (a) *Minimal rights.* The first restriction traces a path which follows the vertical axis until the point $(0, c_2)$, then follows a horizontal line until the claims vector; the second restriction traces a path which follows the horizontal axis until the point $(c_1, 0)$, then follows a vertical line until the claims vector. For each rule, its paths of awards lie inside the paths of acceptable vectors of *minimal rights*, hence all rules satisfy the bound. (b) *Reasonable lower bound.* The first restriction traces a path which follows the 45° line until the point $(\frac{c_1}{2}, \frac{c_1}{2})$, then follows a vertical line until the point $(\frac{c_1}{2}, c_2 + \frac{c_1}{2})$; the second restriction traces a path which follows the 45° line until the point $(\frac{c_2}{2}, \frac{c_2}{2})$, then follows a horizontal line until the point $(c_1 + \frac{c_2}{2}, \frac{c_2}{2},)$. If $c_1 \neq c_2$, the paths of awards (for c) of the *proportional* and the *constrained equal losses* rules lie outside the path of acceptable vectors (for c) of *reasonable lower bound*, hence they fail the bound. (c) *Min lower bound.* Both restrictions trace out a path which follows the 45° line until the point $(\frac{\min\{c_i\}}{2}, \frac{\min\{c_i\}}{2})$, then the first restriction traces a vertical line until the point $(\frac{\min\{c_i\}}{2}, c_1 + c_2 - \frac{\min\{c_i\}}{2})$, the second traces the symmetric horizontal line. The *proportional* and the *constrained equal losses* rules fail the bound for the same reason that they fail *reasonable lower bound*.

than the common claim, each agent should get at least $\frac{1}{|N|}$th of the common claim. In the original problem each agent's claim is at least as large as her claim in the reference situation, hence her bound should not decrease. For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, agent i's min lower bound is $\mu_i(c, E) = \frac{1}{|N|} \min\{\{c_j\}_{j \in N}, E\}$. The **min lower bound** is given by:

$$\mu(c, E) = (\mu_i(c, E))_{i \in N}.$$

Before studying properties for lower bounds, we note two intuitive ways to determine which of two lower bounds is stronger. Each defines a partial order on the space of rules. The first one compares, problem-by-problem, the rights vectors of the lower bounds. If for each problem, the rights vector of one lower bound dominates the rights vector of the other, then the former is stronger. Using this comparison, *minimal rights* is neither weaker nor stronger than either *reasonable lower bound* or *min lower bound*, but *min lower bound* is weaker than *reasonable lower bound*. Graphically, this comparison corresponds to containment of the paths of acceptable vectors (see Figure 3).

The second way compares the set of rules satisfying the bound. If the set of rules satisfying one lower bound is contained in the set of rules satisfying the other, then the former is stronger. Using this comparison, *minimal rights* is weaker than *min lower bound* which is weaker than *reasonable lower bound*. In fact, *minimal rights* is vacuously satisfied

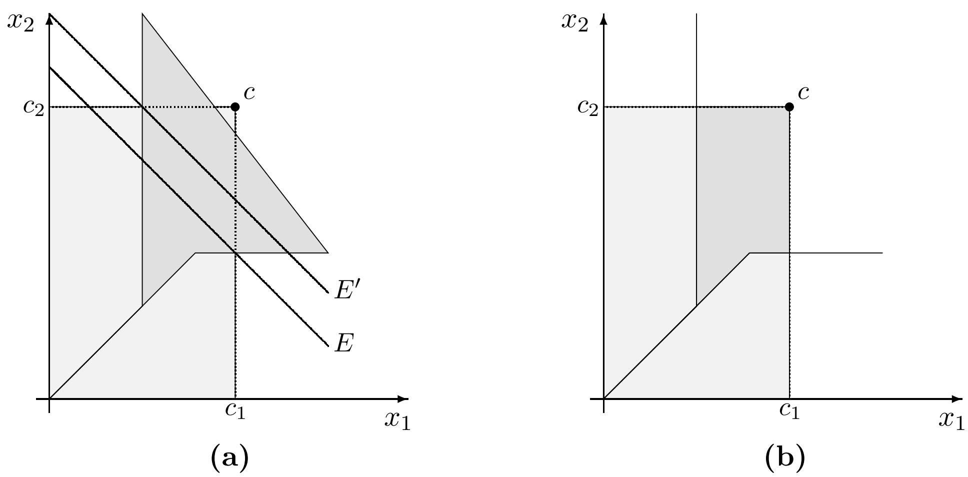

Figure 3: **Comparing minimal rights to reasonable lower bound.** (a) *Problem-by-problem comparison.* The bounds are non-comparable. For endowments smaller than E, the rights vectors of *reasonable lower bound* dominate the rights vectors of *minimal rights*; for endowments larger than E and smaller than E', there is no domination; for endowments greater than E', the rights vectors of *minimal rights* dominate the rights vectors *reasonable lower bound.* (b) *Set of rules satisfying each bound.* The paths of acceptable awards of *reasonable lower bound* are contained in the paths of acceptable awards of *minimal rights.* Hence, *reasonable lower bound* is stronger than *minimal rights.*

by all rules. Graphically, this comparison corresponds to containment of the paths of acceptable vectors restricted to the set of awards vectors. Given a lower bound, its paths of acceptable vectors restricted to the set of awards are its **paths of acceptable awards** (see Figure 3).

3.2 Properties on lower bounds

Many of the following properties are taken from the literature. They were formulated for rules, but are also applicable to lower bounds. Properties can be classified as either punctual or relational. The first apply to individual problems, while the latter relate the rights vectors recommended by the lower bound across problems related in a specific way.

We start with properties that can be interpreted as expressing the idea of horizontal equity. First, agents with equal claims should be assigned equal rights:

Equal treatment of equals: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $\{i, j\} \in N$, if $c_i = c_j$, $b_i(c, E) = b_j(c, E)$.

Minimal rights, *reasonable lower bound*, and *min lower bound* satisfy *equal treatment of equals*. In some situations it is reasonable to treat agents with the same claims differently, and generalizations of equal treatment of equals treating agents according to a vector of weights have been proposed (Hokari and Thomson 2003).

The next property implies *equal treatment of equals*. The rights assigned by a lower bound should not depend on the names of the agents. The lower bounds introduced earlier satisfy this property. Denote by Π^N the set of all permutations on N:

Anonymity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each permutation $\pi \in \Pi^N$, $b(\pi(c), E) = \pi(b(c, E))$.

The next property also implies *equal treatment of equals* but it is logically independent of *anonymity*. If agent i's claim is at least as large as agent j's, agent i's right should be at least as large as agent j's (Aumann and Maschler 1985).

Order preservation: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each pair $\{i, j\} \subseteq N$, if $c_i \geq c_j$, $b_i(c, E) \geq b_j(c, E)$.

In the literature, *order preservation* (of awards) is usually imposed together with a *dual* property for the losses of the agents.[10] Given a problem and an awards vector, each agent's loss is given by the difference between her claim and her award. In our setting, a lower bound (on awards) has a *dual lower bound on losses*. Given a lower bound b, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $b^d(c, E) = c - b(c, E)$ defines such a bound. Agents with smaller claims should be assigned smaller lower bounds on losses:[11]

Order preservation of losses: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each pair $\{i, j\} \subseteq N$, if $c_i \geq c_j$, then, $b_i^d(c, E) \geq b_j^d(c, E)$.

When a lower bound satisfies both *order preservation* and *order preservation of losses* we say that it satisfies *full order preservation*. The three bounds presented earlier satisfy *full order preservation*.

We turn to monotonicity properties. They are relational conditions. If an agent's claim increases, her right should not decrease:

Claims monotonicity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, if $c_i < c'_i$, $b_i(c, E) \leq b_i(c'_i, c_{-i}, E)$.[12]

We can apply the monotonicity property to the agents whose claims remain the same. When an agent's claim increases, the other agent's rights should not increase:

Others-oriented claims monotonicity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, if $c_i < c'_i$, $b_{N \setminus \{i\}}(c, E) \geq b_{N \setminus \{i\}}(c'_i, c_{-i}, E)$.

When the endowment increases, agents should not be assigned smaller rights:

Resource monotonicity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $i \in N$, and each $E' < E$, $b(c, E') \leq b(c, E)$.

When there are more agents, the endowment has to be divided among a larger population. An agent's right should not increase:

Population monotonicity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subseteq N$, if $E \leq \sum_{i \in N'} c_i$, $b_{N'}(c, E) \leq b(c_{N'}, E)$.

The three bounds satisfy *claims monotonicity* and *resource monotonicity*. *Min lower bound* does not satisfy *others-oriented claims monotonicity* nor *population monotonicity*, but the other two bounds do.

We turn to operational properties. Since an agent cannot get more than the endowment, the rights vector should not be affected if we truncate the claims at the endowment. All three bounds satisfy the property:

Claims truncation invariance: For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $b(c, E) = b(t(c, E), E)$.

The next property is a variable population invariance property. For rules, it states that we can obtain the awards vector "sequentially", first assigning to some agents their awards, and then distributing the remaining endowment to the remaining agents. Formally, starting with a population $N \in \mathcal{N}$, a population $N' \subset N$, a problem $(c, E) \in \mathcal{C}^N$, and a vector $x \in X(c, E)$, the (N', x)-*reduced problem* is $r^x_{N'}(c, E) = (c_{N'}, \sum_{i \in N'} x_i)$. It defines

[10] For a systematic treatment of duality see Thomson and Yeh (2005).

[11] A smaller lower bound on her losses is better for an agent. *Order preservation of losses* is a way to prevent agents with smaller claims form being treated too harshly.

[12] Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and $N' \subseteq N$. The vector $c_{N'}$ is the vector c restricted to the population N'. The vector c_{-i} is the vector c restricted to the population $N \setminus \{i\}$. The vector $(c_{N \setminus N'}, c'_{N'})$ is the vector c where the N'th coordinates have been replaced by $c'_{N'}$.

a problem where only the agents in the subset N' of the initial population is present, their claims are the same as they were initially, and the endowment is equal to the sum of their awards. A rule φ is **consistent** if for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subseteq N$, $\varphi_{N'}(c, E) = \varphi(r_{N'}^{\varphi(c,E)}(c, E))$. Alternatively we can define the (N', x)-*reduced problem* $r_{N'}^x(c, E) = (c_{N'}, E - \sum_{i \in N \setminus N'} x_i)$. Given the *efficiency* of a rule, both definitions coincide.

For lower bounds, coincidence fails. The endowments for each version of the reduced problem may differ. If we use the first reduction, *consistency* has strong implications on the *efficiency* of the bound.[13] If we use the second reduction, the sum of the claims in the reduced problem may be smaller than the endowment,[14] and the reduced vector may not define a problem.

We could condition the conclusions on whether or not the reduced vector defines a problem. We do not follow this approach, but instead formulate a related invariance property for fixed populations. It states that we can assign to some agents their rights, revise their claims and the endowment accordingly, and apply the lower bound to the new problem; we require that each agent who initially was not assigned her right should be assigned the same right in both problems.

Given a lower bound b, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subseteq N$, the (N', b)-*partially resolved problem* is $p_{N'}^b(c, E) = \Big(((c - b(c, E))_{N \setminus N'}, c_{N'}), E - \sum_{i \in N \setminus N'} b_i(c, E)\Big)$.[15] It defines the problem (c', E') where, (i) for each agent $i \in N \setminus N'$, the claim c_i' is what remains of her initial claim after she is assigned her right, (ii) for each agent $i \in N'$ her claim remains the same, and (iii) E' is what remains of the endowment after assigning their rights to the agents in $N \setminus N'$. When $N' = \emptyset$, all agents are assigned their rights. We call the revised problem the *b-partially resolved problem*, and denote it $p^b(c, E)$ dropping the $\emptyset$ subscript:

Invariance under partial assignments: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subseteq N$, $b_{N'}(c, E) = b_{N'}\left(p_{N'}^b(c, E)\right)$.

Minimal rights satisfies *invariance under partial assignments*, but *reasonable lower bound* and *min lower bound* fail it. Now, we formulate two conditions that are specific to lower bounds. Both of them imply that the total amount assigned should be positive:

Strong positivity: For each $N \in \mathcal{N}$ and each *non-degenerate* problem $(c, E) \in \mathcal{C}^N$, if $c_j > 0$, $b_j(c, E) > 0$.

Positivity: For each $N \in \mathcal{N}$ and each *non-degenerate* problem $(c, E) \in \mathcal{C}^N$, $b(c, E) \geq 0$.

It is clear that *strong positivity* implies *positivity.* Both properties can be seen as protecting agents from receiving nothing. The first one protects each agent separately, the second protects the agents as a group. Since we think of lower bounds as protecting each agent separately, *strong positivity* is of particular interest. When we interpret lower bounds as *inefficient* rules, the properties can be thought of as minimal *efficiency* requirements; a lower bound should assign a positive amount whenever possible.[16] *Minimal rights* and *min lower bound* fail *positivity* (and hence *strong positivity*). *Reasonable lower bound* satisfies

[13]Let $N \in \mathcal{N}$, $N' \subset N$, and $(c, E) \in \mathcal{C}^N$. Consider the reduced problem $r_{N'}^{b(c,E)}(c, E)$, *consistency* implies that $b_{N'}(c, E) = b(r_{N'}^{b(c,E)}(c, E)) = b\left(c_{N'}, \sum_{i \in N'} b_i(c, E)\right)$. Thus, in the reduced problem, the rights vector is *efficient*.

[14]Let $N = \{1, 2\}$, $(c_1, c_2, E) = (1, 3, 2)$, $b(1, 3, 2) = (\frac{1}{2}, \frac{1}{2})$, and $N' = \{1\}$. The reduction leads to the vector $(c_1, E - b_2(1, 3, 2)) = (1, 1.5) \notin \mathcal{C}^{\{1\}}$.

[15]It is easy to see that the set of problems is closed under this operation.

[16]For this interpretation to be valid, we should require that the lower bound assigns a positive amount to each problem with a positive endowment.

strong positivity (and hence *positivity*). In two-agent problems, *min lower bound* satisfies both properties.

When a problem is *degenerate*, there is no conflict among the agents. A lower bound should distribute the endowment fully:

Conditional efficiency: For each $N \in \mathcal{N}$ and each *degenerate* problem $(c, E) \in \mathcal{C}^N$, $b(c, E) = X(c, E)$.

4 Extending a lower bound

Suppose we agree on a lower bound and find that, for some problem, the rights vector of its *partially resolved problem* is positive. In such situations, we can extend the lower bound by adding to the rights vector for the original problem the rights vector of the revised problem. The *recursive extension of a bound* is obtained by repeatedly performing this operation.

Similarly, starting from a rule and a lower bound, we can construct a new rule in the following way: for each problem, first assign to each agent her right, then apply the rule to the revised problem.[17] For some rule-bound pairs the constructed rule coincides with the original rule. We say that the rule is *invariant under the assignment of the bound.*

In this section we show that, if a lower bound satisfies *positivity*, repeatedly assigning the lower bound singles out the unique rule satisfying *invariance under the assignment of the bound.* First, we formally define the recursive extension of a lower bound, and illustrate the process using the lower bounds in Section 3.1. Then, we define *invariance under the assignment of a bound.* Finally, we state and prove our main theorem. We restrict attention to continuous lower bounds. Similar results can be obtained for discontinuous bounds, but the proofs are significantly easier for continuous bounds.

4.1 Recursive extensions

Consider a lower bound and extend it, according to the operation described above, to get a new lower bound. Then, extend the new lower bound using the same operation. The recursive extension is obtained by repeatedly extending a lower bound.

Definition 3. Given a lower bound b, let $b^0 \equiv b$. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $k \in \mathbb{N}$, the **k-extension of b** is the lower bound defined recursively by:

$$b^k(c, E) = b^{k-1}(c, E) + b^{k-1}(p^{b^{k-1}}(c, E)).$$

For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, the sequence $\{b^k(c, E)\}$ is increasing and bounded. Thus it has a limit, and this limit defines the rights vector assigned by the recursive extension of the bound to the problem (c, E).

Definition 4. Given a lower bound b, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, the **recursive extension of b** assigns the rights vector $B(c, E)$, where:

$$B(c, E) = \lim_{k \to \infty} b^k(c, E).$$

[17] It is easy to see that this operation defines a rule. For a detailed study of this operation, with respect to *minimal rights*, see Thomson and Yeh (2005).

If a lower bound is continuous, it is easy to see that its recursive extension is also continuous. The next lemma provides an alternative way of obtaining the recursive extension of a lower bound.

Lemma 1. *Let B be the recursive extension of b. For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $B(c, E)$ satisfies the following two conditions:*[18]

$$B(c,E) = b(c,E) + B\left(p^b(c,E)\right),$$
$$b(c,E) = 0 \;\Rightarrow\; B(c,E) = 0.$$

Moreover, if b is continuous, the recursive extension of B is the unique lower bound satisfying the conditions above.

We call the first condition the *recursive* condition and the second one the *no start* condition.

Proof. The fact that the recursive extension satisfies the conditions is straightforward. Conversely, we show that if b is continuous, it is the unique lower bound satisfying them. Let B and B' satisfy both conditions. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Applying the *recursive* condition to (c, E), we get $B(c, E) - B'(c, E) = B(p^b(c, E)) - B'(p^b(c, E))$. Recursively applying the argument, we construct a bounded and decreasing sequence of problems $(c^k, E^k) = p^b(c^{k-1}, E^{k-1})$. This sequence has a limit $(\underline{c}, \underline{E})$ and, by continuity of B and B',[19] $B(c, E) - B'(c, E) = B(\underline{c}, \underline{E}) - B'(\underline{c}, \underline{E})$. We call the last equality condition $(*)$. Since the sequence is convergent, $b(c^{k-1}, E^{k-1}) \to 0$ and, by continuity of b, $b(\underline{c}, \underline{E}) = 0$. Hence, by *no start* $B(\underline{c}, \underline{E}) = 0 = B'(\underline{c}, \underline{E})$. Substituting in condition $(*)$ we get $B(c, E) = B'(c, E)$. Since this was done for an arbitrary $N \in \mathcal{N}$ and an arbitrary $(c, E) \in \mathcal{C}^N$, $B = B'$. Thus, there is a unique rule satisfying both conditions. $\square$

Even though the recursive extension of a lower bound is unique, there may be several lower bounds who share the same recursive extension. In particular, as Corollary 1 shows, if a bound is continuous, then its recursive extension is also its own recursive extension.

Lemma 2. *Let b be a continuous lower bound and B its recursive extension. For each $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$, $B(p^B(c, E)) = 0$ and $b(p^B(c, E)) = 0$.*

Proof. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. By continuity, $b(p^B(c, E)) = \lim_{k\to\infty} b(p^{b^k}(c, E))$. By construction of the recursive extensions, $\{b^k(c, E)\}_{k\in\mathbb{N}}$, $\lim_{k\to\infty} b(p^{b^k}(c, E) = 0$. Hence $b(p^B(c, E)) = 0$ and, by *no start*, $B(p^B(c, E)) = 0$. $\square$

Corollary 1. *Let b be a continuous lower bound and B its recursive extension. Then, B is its own recursive extension.*

Proof. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. By Lemma 2, $B(p^B(c, E)) = 0$. Hence, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $k \in N$, $B^k(c, E) = B(c, E)$, where B^k is the k-extension of B. Hence B is its own recursive extension. $\square$

To illustrate the recursive extension of a bound, we construct the extensions of *minimal rights* and *min lower bound*. For *reasonable lower bound* we refer to Dominguez and Thomson (2006) for a detailed treatment.

[18] We abuse notation and write $x = 0$ for $(x_1, ..., x_n) = (0, ..., 0)$.

[19] Recall that if a lower bound b is continuous, then its recursive extension B is also continuous.

Proposition 1. *The recursive extension of minimal rights is minimal rights itself.*

We use the fact that, for each problem, after assigning *minimal rights* and revising the problem accordingly, *minimal rights* of the revised problem are equal to 0 (Thomson 2005a), that is, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $m(p^m(c, E)) = 0$.

Proof. Let M be the recursive extension of *minimal rights*. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. By the *recursive* condition, $M(c, E) = m(c, E) + M(p^m(c, E))$. Since $m(p^m(c, E)) = 0$, by *no start*, $M(p^m(c, E)) = 0$. Hence, $M(c, E) = m(c, E)$. □

The proof of Proposition 1 illustrates the use of *no start*. Without it, we cannot guarantee that the recursive extension of the revised problem is 0. Consider the following (discontinuous) bound: let $N = \{1, 2\}$, $\bar{c} = (2, 3)$, $\bar{E} = 1$, and let $A = \{(c, E) \in \mathcal{C}^N \mid p^m(c, E) = (\bar{c}, \bar{E})\}$. Define the bound M' as follows:

$$\begin{aligned} &\text{for each } (c, E) \notin A, \quad M'(c, E) = m(c, E), \\ &\text{for each } (c, E) \in A, \quad M'(c, E) = m(c, E) + (.5, .5). \end{aligned}$$

M' satisfies the *recursive* condition with respect to *minimal rights*, but fails *no start*.

The recursive extension of *min lower bound* does not coincide with itself. It assigns to each agent, the minimum of the smallest claim and $\frac{1}{|N|}$th of the endowment. Several rules satisfy this extension, but for two-agent problems they coincide with the *constrained equal awards rule* introduced in Section 3.1.

Proposition 2. *The recursive extension of min lower bound assigns, to each agent, the common amount* $\lambda \in \mathbb{R}_+$, *such that* $\lambda = \min\{\{c_i\}_{i \in N}, \frac{E}{|N|}\}$.

Proof. The fact that all agents are assigned a common amount λ is a straightforward implication of the fact that, for each problem, *min lower bound* assigns the same rights to all agents. To prove that $\lambda = \min\{\{c_i\}_{i \in N}, \frac{E}{|N|}\}$ we note that, by *claims boundedness* and *feasibility*, $\lambda \leq \min\{\{c_i\}_{i \in N}, \frac{E}{|N|}\}$.

Let ν be the recursive extension of *min lower bound*. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Since each agent is assigned λ, $(p^\nu(c, E)) = ((c_i - \lambda)_{i \in N}, E - |N|\lambda)$. If $\lambda < \min\{\{c_i\}_{i \in N}, \frac{E}{|N|}\}$, then $\frac{1}{|N|} \min\{\{c'_i\}_{i \in N}, E'\} > 0$, where $(c', E') = p^\nu(c, E)$. Hence, $\mu(p^\nu(c, E)) > 0$, contradicting the conclusion of Lemma 2. □

To see that, for two-agent populations, all rules satisfying the recursive extension of *min lower bound* coincide with the *constrained equal awards rule*, consider a problem $(c, E) \in \mathcal{C}^{\{1,2\}}$. Suppose, without loss of generality, that $c_1 < c_2$. If $E \leq 2c_1$ the rights vector is *efficient*, and any rule satisfying the bound divides the endowment equally. If $E > 2c_1$ the bound assigns to each agent c_1, thus any rule satisfying the bound fully honors agent 1's claim and, by *efficiency*, agent 2 gets the remainder. Thus, for each two-agent problem, the awards vector assigned by a rule satisfying the recursive extension of *min lower bound* is equal to the awards vector assigned by the *constrained equal awards rule* (see Figure 4).

Many rules satisfy the recursive extension of *min lower bound*. The *constrained equal awards rule* does, but as we will see in Section 6 it is the only *consistent* rule satisfying this bound.

One may ask when extending a lower bound singles out a rule. The complete answer is given by Theorem 1. For now, we note that *minimal rights* fails *positivity* and so does *min lower bound*, although, in two-agent problems, the latter satisfies it.

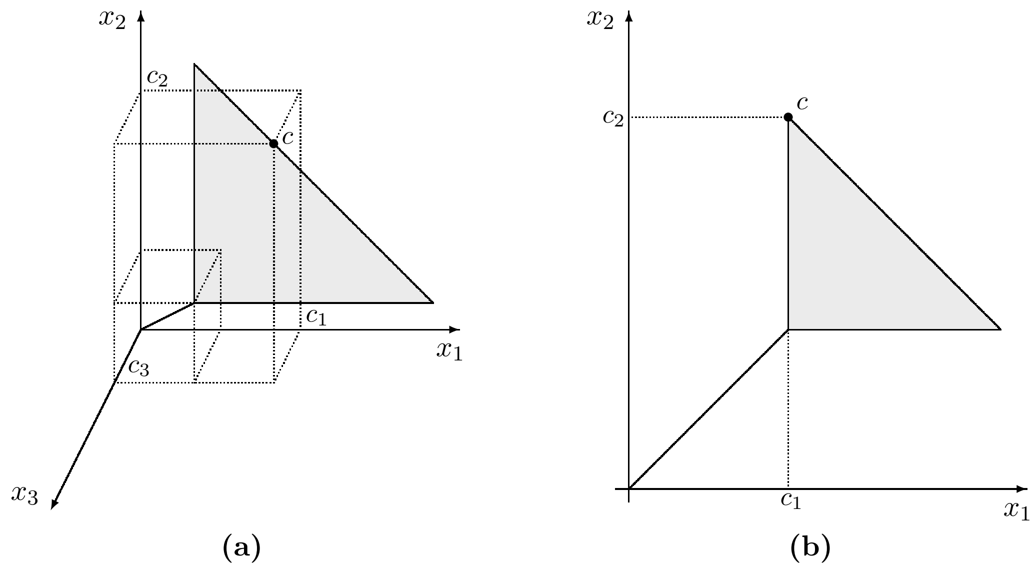

Figure 4: **Graphical representation of the recursive extension of *min lower bound*.** (a) Path of acceptable vectors. Many rules satisfy the recursive extension of *min lower bound*. (b) Path of acceptable vectors. When $|N| = 2$, all rules satisfying the bound coincide with the *constrained equal awards rule*.

4.2 An invariance requirement

Given a lower bound, the invariance requirement that a rule should be obtainable in two ways: (i) applying the rule directly or (ii) first assigning the rights vector and then applying the rule to a revised problem, was introduced to the literature with respect to *minimal rights* (Curiel et al. 1987). For two-agent problems, along with *equal treatment of equals* and *claims truncation invariance*, this property characterizes a rule known as the *contested garment rule* (Dagan 1996). We apply the idea to any lower bound:

Definition 5. Given a lower bound b, a rule φ satisfies **invariance under the assignment of b**, if for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$,

$$\varphi(c, E) = b(c, E) + \varphi(p^b(c, E)).$$

Many rules satisfy *invariance under the attribution of minimal rights*, but only one rule satisfies *invariance under the attribution of reasonable lower bound* (Dominguez and Thomson 2006). One may ask when this invariance property yields a unique rule. Again, Theorem 1 provides the complete answer. For now, we note that the first bound fails *positivity* while the latter satisfies it.

To prove the main theorem, we need the following lemmas:

Lemma 3. *Let B be the recursive extension of b. For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $p^B(c, E) = p^B(p^b(c, E))$.*

Proof. Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$. Recall that, by the *recursive* condition, the recursive extension satisfies $B(c,E) = b(c,E) + B(p^b(c,E))$. Then,

$$\begin{aligned} p^B(c,E) &= \left(c - B(c,E), E - \sum_{i\in N} B_i(c,E)\right) \\ &= \left(c - b(c,E) - B\left(p^b(c,E)\right), E - \sum_{i\in N} b_i(c,E) - \sum_{i\in N} B_i\left(p^b(c,E)\right)\right) \\ &= \left(p^b(c,E) - B\left(p^b(c,E)\right), (E - \sum_{i\in N} b_i(c,E)) - \sum_{i\in N} B_i\left(p^b(c,E)\right)\right) \\ &= p^B\left(p^b(c,E)\right). \end{aligned}$$

□

The next lemma provides a useful way to explain what it means for there to be a unique rule satisfying a lower bound. If there is a unique rule satisfying a bound, then starting from any problem and assigning the rights vector recommended by the bound, the *partially resolved problem* is *degenerate.*

Lemma 4. *Let b be a lower bound, the following statements are equivalent*

(i) There is a unique rule satisfying b.

(ii) For each $N \in \mathcal{N}$ and each $(c,E) \in \mathcal{C}^N$, $|X(p^b(c,E))| = 1$.

Proof. To prove that (i) implies (ii) assume that there exist $N \in \mathcal{N}$ and $(c^*, E^*) \in \mathcal{C}^N$ such that $|X(p^b(c^*,E^*))| > 1$. We show that at least two rules satisfy b. Let φ satisfy b, $y = \varphi(c^*,E^*)$, and $x = (y - b(c^*,E^*))$. Then, $x \in X(p^b(c^*,E^*))$. Let $x' \in X(p^b(c^*,E^*))$ be such that $x' \neq x$, and $y' = x' + b(c^*,E^*)$. Then, $y' \neq y$ and $y' \in X(c^*,E^*)$. Let $\phi(c^*,E^*) = y'$, and, for each $(c,E) \neq (c^*,E^*)$, let $\phi(c,E) = \varphi(c,E)$. Then, $\phi \neq \varphi$ and both φ and ϕ satisfy b.

Now we prove that, if for each $N \in \mathcal{N}$ and each $(c,E) \in \mathcal{C}^N$, $|X(p^b(c,E))| = 1$, then there is a unique rule satisfying b. Let $N \in \mathcal{N}$, $(c,E) \in \mathcal{C}^N$, φ a rule satisfying b, and $x = \varphi(c,E) - b(c,E)$. By *claims boundedness*, $x \leq c - b(c,E)$. Since φ satisfies b, $x \geq 0$. Summing over the agents, $\sum_{i\in N} x_i = \sum_{i\in N} \varphi_i(c,E) - \sum_{i\in N} B_i(c,E) = E - \sum_{i\in N} B_i(c,E)$. These three conditions imply that $x \in X(p^b(c,E))$. Since $|X(p^b(c,E)| = 1$, $x = X(p^b(c,E))$. Thus, $\varphi(c,E) = b(c,E) + X(p^b(c,E))$. □

Corollary 2. *Let b be a lower bound and φ the unique rule satisfying it. For each $N \in \mathcal{N}$ and each $(c,E) \in \mathcal{C}^N$,*

$$\varphi(c,E) = b(c,E) + X(p^b(c,E).$$

Proof. Direct implication of the proof of Lemma 4. □

We are now ready for the main theorem:

Theorem 1. *Let b be a continuous lower bound. The following conditions are equivalent:*

(i) b satisfies positivity.

(ii) There is a unique rule satisfying the recursive extension of b.

(iii) There is a unique rule satisfying invariance under the assignment of b.

Moreover, the rules in (ii) and (iii) coincide.

Proof. First we show that (i) implies (ii). Let B be the recursive extension of b. Let $N \in \mathcal{N}$, and $(c, E) \in \mathcal{C}^N$. By Lemma 2, $b(p^B(c, E)) = 0$. Hence, by *positivity*, $p^B(c, E)$ is *degenerate*, and therefore $|X(p^B(c, E))| = 1$. Since the argument holds for an arbitrary $N \in \mathcal{N}$ and an arbitrary $(c, E) \in \mathcal{C}^N$, by Lemma 4, there is a unique rule satisfying B.

Now we prove that (ii) implies (iii), and that both rules coincide. Let B be the recursive extension of b and φ the unique rule satisfying B. First we show that φ satisfies *invariance under the assignment of b*. Then, we show that no other rule does.

Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Since $\varphi(c, E)$ is the unique rule satisfying B, by Corollary 2, $\varphi(c, E) = B(c, E) + X(p^B(c, E))$. By the *recursive* condition $\varphi(c, E) = b(c, E) + B(p^b(c, E)) + X(p^B(c, E))$. By Lemma 3, $\varphi(c, E) = b(c, E) + B(p^b(c, E)) + X(p^B(p^b(c, E)))$. By Corollary 2 applied to the problem $p^b(c, E)$, $\varphi(c, E) = b(c, E) + \varphi(p^b(c, E))$. Thus, φ satisfies *invariance under the assignment of b*.

Now we prove that no other rule satisfies *invariance under the assignment of b*. Let B be the recursive extension of b, φ the unique rule satisfying B, and ϕ a rule satisfying *invariance under the assignment of b*. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Since ϕ satisfies *invariance under the assignment of b*, $\phi(c, E) = b(c, E) + \phi(p^b(c, E)$. Repeatedly applying *invariance under the assignment of b*, we get $\phi(c, E) = B(c, E) + \phi(p^B(c, E))$. By Lemma 4, $|X(p^B(c, E))| = 1$. Thus, $\phi(c, E) = B(c, E) + X(p^B(c, E))$. By Corollary 2, $\phi(c, E) = \varphi(c, E)$. Since the argument holds for an arbitrary $N \in \mathcal{N}$ and an arbitrary $(c, E) \in \mathcal{C}^N$, $\phi = \varphi$.

Finally we show that (iii) implies (i). Let φ be the unique rule satisfying *invariance under the assignment of b*. Let $N \in \mathcal{N}$ and $(\bar{c}, \bar{E}) \in \mathcal{C}^N$ be *non-degenerate*. We show that $b(\bar{c}, \bar{E}) \geq 0$.

Suppose, by contradiction, that $b(\bar{c}, \bar{E}) = 0$. Let $x \in X(\bar{c}, \bar{E})$ such that $x \neq \varphi(\bar{c}, \bar{E})$, it exists since $(\bar{c}, \bar{E})$ is *non-degenerate*. Let $A^0 = \{(\bar{c}, \bar{E})\}$, for each $k \geq 1$, let $A^k = \{(c, E) \mid p^b(c, E) \in A^{k-1}\}$. It is easy to show that $A^{k-1} \subseteq A^k$. Let $A \subseteq \mathcal{C}^N$ be the smallest set such that, for each $k \in \mathbb{N}$, $A^k \subseteq A$. For each $(c, E) \in \mathcal{C}^N$ define the rule ϕ by:

$$(c, E) \in A \Rightarrow \phi(c, E) = b(c, E) + x,$$

$$(c, E) \in A^c \Rightarrow \phi(c, E) = \varphi(c, E).^{20}$$

Then, $\phi \neq \varphi$ and ϕ satisfies *invariance under the attribution of b*, a contradiction with uniqueness of such rule. Thus, $b(\bar{c}, \bar{E}) \geq 0$. □

It is worth noting that, if a lower bound fails *positivity*, there are rules satisfying its recursive extension, but failing *invariance under the assignment* of the bound. For example, consider *minimal rights*. As we have seen, *minimal rights* is its own recursive extension (Proposition 1). Moreover, all rules satisfy *minimal rights*. Hence, all rules satisfy the recursive extension of *minimal rights*. On the other hand, many rules fail *invariance under the assignment of minimal rights*. One such rule is the *proportional* rule. This example shows that, if we drop *positivity*, there is no equivalence between satisfying the recursive extension of a bound and satisfying *invariance under the assignment of the bound*.

[20] A complete definition requires defining ϕ for all populations. For each $N' \in \mathcal{N}$, such that $N' \neq N$, define ϕ to coincide with φ.

A straightforward corollary of Theorem 1 is that there is a unique rule satisfying *invariance under the assignment of reasonable lower bound* (Dominguez and Thomson 2006). By the following corollary, we can also conclude that the recursive extension of *reasonable lower bound* is such a rule.

Corollary 3. *If a continuous lower bound b satisfies positivity and conditional efficiency, then its recursive extension is efficient. Thus, the recursive extension of b defines the unique rule satisfying invariance under the assignment of b.*

Proof. Let b satisfy *positivity* and *conditional efficiency*, B its recursive extension, and φ a rule satisfying B. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. We show that $B(c, E) = \varphi(c, E)$. By Lemma 2, $b(p^B(c, E)) = 0$. Hence, by *positivity*, $|X(p^B(c, E))| = 1$. By *conditional efficiency*, $b(p^B(c, E)) = X(p^B(c, E))$. Thus, $B(p^B(c, E)) = X(p^B(c, E))$. By Lemma 2, $B(p^B(c, E)) = 0$. Thus, $X(p^B(c, E)) = 0$. By Theorem 1, $\varphi(c, E) = B(c, E) + X(p^B(c, E))$. Thus, $B(c, E) = \varphi(c, E)$. □

This result can also be interpreted in the following way: suppose that a lower bound satisfies *conditional efficiency*. Then, its recursive extension "inherits" *conditional efficiency*. Moreover, if the bound also satisfies *positivity*, its recursive extension "inherits" *efficiency*. In the next section we ask this sort of questions for the properties introduced in Section 3.

A converse statement of Corollary 3 holds by weakening *conditional efficiency* in the following way: Let $\varepsilon > 0$. A lower bound b satisfies **ε-conditional efficiency** if, for each $N \in \mathcal{N}$ and each *degenerate* problem $(c, E) \in \mathcal{C}^N$, $\sum_{i \in N} b_i(c, E) \geq \varepsilon E$. Clearly, if $\varepsilon > 1$ the condition cannot be satisfied. If $\varepsilon = 1$ the condition corresponds to *conditional efficiency*. If $\varepsilon < 1$ the condition is weaker than *conditional efficiency*. It states that, for a *degenerate* problem with a positive endowment, the agent with a positive claim should receive a positive right.

Corollary 4. *Let b be a continuous lower bound and $\varepsilon > 0$. If b satisfies ε-conditional efficiency and positivity, then its recursive extension is efficient. Conversely, if the recursive extension of b is efficient, then for some $\varepsilon > 0$, b satisfies ε-conditional efficiency and positivity.*

Proof. The proof of the first statement follows the same logic as Corollary 3. We prove the converse statement. Let B be the recursive extension of b. If B is *efficient*, there is a unique rule satisfying B. By Theorem 1, b satisfies *positivity*. Assume, by contradiction, that for each $\varepsilon > 0$, b fails *ε-conditional efficiency*. Then, there is $N \in \mathcal{N}$ and a *degenerate* problem $(c, E) \in \mathcal{C}^N$ such that $b(c, E) = 0$ and $X(c, E) \geq 0$.[21] By *no start*, $B(c, E) = 0$. Hence, B is not *efficient*, contradicting our initial hypothesis. Thus, there is $\varepsilon > 0$ such that b satisfies *ε-conditional efficiency*. □

5 Inheritance of properties

When a lower bound satisfies a property its recursive extension may or may not satisfy the property. For some properties, if a lower bound satisfies them, they are automatically satisfied by its recursive extension. In that case, we say that the property is "inherited" by the recursive extension. Some properties are inherited on their own (directly); other

[21] The existence of a *degenerate* problem (c, E) with $b(c, E) = 0$ is implied by the closedness of the set of *degenerate* problems and the continuity of b.

properties, which are not directly inherited, are inherited if they are imposed together with other properties (assisted). In this section we undertake a systematic investigation of inheritance of properties. Some proofs are straightforward and we do not provide them. We relegate most examples for negative inheritance results to the appendix. It is worth noting that the properties are inherited by the recursive extension, and not necessarily by rules satisfying the recursive extension. When a lower bound satisfies positivity, the unique rule satisfying the recursive extension usually inherits properties, but we do not have a formal proof for this statement.[22]

Definition 6. A property is **inherited** (by the recursive extension of a bound) if, whenever a bound satisfies the property, its recursive extension also satisfies it.

We begin with the three horizontal equity properties. For *equal treatment of equals*, note that, for each problem, the bound assigns equal rights to agents with equal claims. In the revised problem their claims are equal and they are assigned equal rights. Thus, the revised bound satisfies *equal treatment of equals*. A recursive argument shows that *equal treatments of equals* is inherited. The same logic is used to prove inheritance of most properties: we start with a problem that satisfies the hypotheses of the property, assign the rights vector and revise the problem accordingly; we check whether the conclusions of the property imply that the revised problem satisfies its hypotheses. If it does, we add the rights vector of the revised problem to the rights vector of the original problem, and check if the revised bound satisfies the conclusions of the property. If it does, a recursive argument proves inheritance of the property.

It is easy to see that *anonymity* is inherited. As the following example shows, *order preservation* is not:

Let $N = \{1, 2\}$. Consider the bound b defined as follows: for the problem $(\bar{c}_1, \bar{c}_2, \bar{E}) = (2, 2.5, 2.5)$, $b(2, 2.5, 2.5) = (0, 1)$. For the problem $(\bar{c}'_1, \bar{c}'_2, \bar{E}') = (2, 1.5, 1.5)$, $b(2, 1.5, 1.5) = (1.5, 0)$. For each other problem $(c, E) \in \mathcal{C}^{\{1,2\}}$, $b(c, E) = 0$.[23] The recursive extension of the bound is: $B(2, 2.5, 2.5) = (1.5, 1)$, $B(2, 1.5, 1.5) = (1.5, 0)$, and for each other problem $(c, E) \in \mathcal{C}^{\{1,2\}}$, $B(c, E) = 0$.[24] Then, b satisfies *order preservation* and B fails it. Hence, the property is not inherited. A dual example shows that *order preservation of losses* is not inherited. Given that the example is not an intuitive lower bound and, in particular, fails *order preservation of losses*, one may wonder if there are well-behaved bounds for which the property is not inherited. The next proposition shows that this is not the case.

Proposition 3. *Full order preservation is inherited.*

Proof. Let b satisfy *full order preservation*. For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, let $b'(c, E) = b(c, E) + b(p^b(c, E))$. We show that b' satisfies *full order preservation*. Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and $\{i, j\} \subseteq N$. Without loss of generality assume $c_i < c_j$. By *full order preservation*, $b_i(c, E) < b_j(c, E)$ and $c_i - b_i(c, E) < c_j - b_j(c, E)$. Consider the problem $(c', E') = p^b(c, E)$. By *order preservation of losses*, $c'_i < c'_j$. By *full order preservation*, $b_i(c', E') < b_j(c', E')$ and $c'_i - b_i(c', E') < c'_j - b_j(c', E')$. Then, $b'_i(c, E) < b'_j(c, E)$ and

[22]By Corollary 4, if for some $\varepsilon > 0$ a lower bound satisfies *ε-conditional efficiency*, then its recursive extension defines a rule and it inherits the properties.

[23]A complete definition requires defining b for all populations. For each $N' \in \mathcal{N}$ such that $N' \neq \{1, 2\}$, and each $(c, E) \in \mathcal{C}^{N'}$, $b(c, E) = 0$.

[24]For the complete definition of B, for each $N' \in \mathcal{N}$ such that $N' \neq \{1, 2\}$, and each $(c, E) \in \mathcal{C}^{N'}$, $B(c, E) = 0$.

$c_i - b'_i(c,E) < c_j - b'_j(c,E)$. Hence, b' satisfies *full order preservation*. A recursive argument shows that *full order preservation* is inherited. $\square$

Monotonicity properties are not directly inherited. In the Appendix we provide examples of bounds to show these negative results. The intuition for the failures is the following: start with a pair of problems (c,E) and $(\bar{c},\bar{E})$ related as in the hypotheses of the property; assign the rights vector and revise the problems accordingly; since, for each problem, the claims and the endowment are revised, the resulting pair of problems typically fail the monotonicity hypotheses. In this case, the lower bound can assign a rights vector that fails the monotonicity conclusions for the original problem, and when summing the rights vectors of the two problems, the resulting revised bound can fail the monotonicity conclusions.

An interesting question is whether or not monotonicity properties are inherited when several of them are imposed at the same time, by bounding the gains and losses from changes in the data,[25] or when dual monotonicity properties on the losses are imposed. As we saw for *order preservation*, its dual property helped inheritance. Moreover, as with most dual properties, dual monotonicity properties have intuitive appeal. We conjecture that, if "enough monotonicity" is imposed, a positive answer can be obtained. In fact, for *reasonable lower bound*, positive answers for inheritance of several monotonicity properties exist (Dominguez and Thomson 2006).

We now turn to *invariance under claims truncation*, we have a positive answer:

Proposition 4. *Invariance under claims truncation is inherited.*

Proof. Let b satisfy *invariance under claims truncation*. Let $N \in \mathcal{N}$ and $(c,E) \in \mathcal{C}^N$. Let $\bar{c} = t(c,E)$ denote the vector of truncated claims and $(\bar{c},\bar{E}) = (t(c,E),E)$ the truncated problem. By *invariance under claims truncation*,

$$b(c,E) = b(\bar{c},\bar{E}). \tag{1}$$

Let $(c',E') = p^b(c,E)$ and $(\bar{c}',\bar{E}') = p^b(t(c,E),E)$ denote the *b-partially resolved problems*. We claim that $(t(c',E'),E') = \big(t(\bar{c}',\bar{E}'),\bar{E}'\big)$. Hence, by *invariance under claims truncation* applied to both problems,

$$b(c',E') = b(\bar{c}',\bar{E}'). \tag{2}$$

To prove the claim, we first note that, by definition, $E = \bar{E}$. Hence, by (1), $E' = \bar{E}'$. Now we show that for each $i \in N$, $t_i(c',E') = t_i(\bar{c}',\bar{E}')$. Let $i \in N$. We distinguish two cases:

i. If $c_i \le E$, then $\bar{c}_i = c_i$ and $c_i - b_i(c,E) = \bar{c}_i - b_i(c,E)$. Using (1), $c_i - b_i(c,E) = \bar{c}_i - b_i(\bar{c},\bar{E})$. Hence, $c'_i = \bar{c}'_i$, and since $E' = \bar{E}'$, by definition of the truncation operator, $t_i(c',E') = t_i(\bar{c}',\bar{E}')$.

ii. If $E < c_i$, then $E - \sum_{j\in N} b_j(c,E) < c_i - b_i(c,E)$. Hence, $E' < c'_i$ and $t_i(c',E') = E'$. We call the last equality condition $(*)$. Moreover, $E < c_i$ implies $\bar{c}_i = E$. Since $E = \bar{E}$, we have $\bar{c}_i - b_i(\bar{c},\bar{E}) = \bar{E} - b_i(\bar{c},\bar{E}) \ge \bar{E} - \sum_{j\in N} b_j(\bar{c},\bar{E})$. Hence, $\bar{c}'_i \ge \bar{E}'$ and $t_i(\bar{c}',\bar{E}') = \bar{E}'$. By condition $(*)$ and since $E' = \bar{E}'$, $t_i(\bar{c}',\bar{E}') = t_i(c',E')$.

[25] Bounds on the gains and losses from changes in the data are intuitive. For instance, when an agents' claim increases, we could require that her right does not increase by more than the increase in her claim. Some bounds on gains and losses can be found in Thomson (2005a).

This completes the proof of the claim.

Now, summing equations (1) and (2),

$$b'(c,E) = b(c,E) + b(c',E') = b(\bar{c},\bar{E}) + b(\bar{c}',\bar{E}') = b'(t(c,E),E).$$

Hence b' satisfies *invariance under claims truncation*. A recursive argument shows that *invariance under claims truncation* is inherited. $\square$

A direct corollary of Proposition 4 is that the unique rule satisfying *reasonable lower bound* satisfies *invariance under claims truncation* (Dominguez and Thomson 2006).

Finally we turn to *invariance under partial assignments*. Unfortunately, it is not inherited. In the appendix we provide an example of a lower bound satisfying the property, but its recursive extension fails it.

6 New characterizations of two classical rules

This section provides new characterizations of the *constrained equal awards rule* (CEA), and the *constrained equal losses rule* (CEL) on the basis of *invariance under the assignment of a lower bound* and *consistency*. For CEA the lower bound imposed is *min lower bound*. For CEL we impose a two-agent bound based on *full order preservation*. To prove these results, we use a result known in the theory of fair allocation as the *Elevator Lemma*.[26] It states that if a *consistent* rule coincides, for two-agent problems, with a "conversely consistent" rule, then coincidence happens in general. For claims problems, Thomson (2005b) formulates a powerful geometric approach for extending two-agent rules to general populations in a *consistent* way (whenever possible); we do not follow that approach but it can also be used to prove our theorems.

We start with the definition of converse consistency. Roughly, it states that if an awards vector is desirable for all subpopulations, then it is desirable for the whole population:

Converse consistency of φ: For each $N \in \mathcal{N}$, each $(c,E) \in \mathcal{C}^N$, and each $x \in X(c,E)$, if for each $N' \subseteq N$ with $|N'| = 2$, we have $x_{N'} = \varphi(c_{N'}, \sum_{i \in N'} x_i)$, then $x = \varphi(c,E)$.

We state the Elevator Lemma and provide a proof:

Lemma 5. *Let φ be a conversely consistent rule, ϕ a consistent rule. If for each $N' \subseteq N$ with $|N'| = 2$, and each $(c',E') \in \mathcal{C}^N$, we have $\varphi(c',E') = \phi(c',E')$. Then, for each $N \in \mathcal{N}$ and each $(c,E) \in \mathcal{C}^N$, $\varphi(c,E) = \phi(c,E)$.*

Proof. Let $N \in \mathcal{N}$ with $|N| > 2$, let $(c,E) \in \mathcal{C}^N$, and $x = \phi(c,E)$. Since ϕ is *consistent*, for each $N' \subset N$ with $|N'| = 2$, $x_{N'} = \phi(c_{N'}, \sum_{i \in N'} x_i)$. Since φ and ϕ coincide for two-agent populations, $x_{N'} = \varphi(c_{N'}, \sum_{i \in N'} x_i)$. Since φ is *conversely consistent*, $x = \varphi(c,E)$. $\square$

6.1 The *constrained equal awards rule*

In Section 3.1 we introduced the *constrained equal awards rule*. It is well-known that it satisfies *consistency* and *converse consistency* (Aumann and Maschler 1985). We are ready to state our next theorem:

Theorem 2. *The constrained equal awards rule is the only rule satisfying consistency and invariance under the assignment of min lower bound.*

[26] The Elevator Lemma is applicable to many other models, for a detailed treatment see Thomson (2004).

Proof. Let φ be a rule satisfying *invariance under the attribution of min lower bound*. By Theorem 1, for two-agent problems φ is the unique rule satisfying the recursive extension of *min lower bound*. By Proposition 2, for two-agent problems, such rule coincides with the *constrained equal awards rule*. The *constrained equal awards rule* is *conversely consistent*, and φ is *consistent*. Then, by the Elevator Lemma, for each $N \in \mathcal{N}$, and each $(c, E) \in \mathcal{C}^N$, $\varphi(c, E) = CEA(c, E)$. □

6.2 The *constrained equal losses rule*

In Section 3.1 we introduced the *constrained equal losses rule*. It is well-known that it satisfies *consistency* and *converse consistency* (Aumann and Maschler 1985). Now we introduce a new bound defined for two-agent problems, it is based on *order preservation* and *order preservation of losses*.

Let (c, E) be a two-agent problem. Without loss of generality assume $c_1 \leq c_2$. *Order preservation* implies that the award of agent 1 should be no greater than the award of agent 2, that is, $x_1 \leq x_2$. We also know that the sum of the awards is equal to the endowment, that is, $x_1 + x_2 = E$. Substituting x_1 into the *order preservation* restriction we get $x_2 \geq \frac{E}{2}$. Similarly, using *order preservation of losses*, we get $x_1 \geq \frac{E-(c_2-c_1)}{2}$. The right hand side of this restriction can be negative, and since in the definition of a lower bound we require rights to be non-negative, *order preservation of losses* requires $x_1 \geq \max\{0, \frac{E-(c_2-c_1)}{2}\}$. This idea leads to the following two-agent lower bound:

Definition 7. Let $N \in \mathcal{N}$ with $|N| = 2$, and $(c, E) \in \mathcal{C}^N$, *agent i's order preservation lower bound* is:

$$op_i(c, E) = \begin{cases} \frac{E}{2} & \text{if } c_{-i} \leq c_i \\ \max\{0, \frac{E-(c_{-i}-c_i)}{2}\} & \text{if } c_{-i} \geq c_i \end{cases}$$

The **order preservation lower bound** is given by:

$$op(c, E) = (op_i(c, E))_{i \in N}.$$

For larger populations it is impossible to write each restriction in terms of the primitives of the model, some of the restrictions will depend on the awards of the other agents. Thus, the *order preservation* restrictions do not define a lower bound.[27]

The next proposition states that the two-agent recursive extension of the *order preservation lower bound* coincides with the (two-agent) *constrained equal losses rule*.

Proposition 5. *Let op be the order preservation lower bound and OP its recursive extension. For each $N \in \mathcal{N}$ with $|N| = 2$, and each $(c, E) \in \mathcal{C}^N$, $OP(c, E) = CEL(c, E)$.*

Proof. By Corollary 4, for two-agent problems, the recursive extension of *order preservation lower bound* defines a rule. We first show that, for each two-agent problem (c, E) with $E \leq |c_2 - c_1|$, the recursive extension of *order preservation lower bound* assigns the entire endowment to the agent with the largest claim (or divides the endowment equally if the claims are equal), and therefore $OP(c, E) = CEL(c, E)$.

Let $N \in \mathcal{N}$ with $|N| = 2$, and $(c, E) \in \mathcal{C}^N$ with $E \leq |c_2 - c_1|$. If $c_1 = c_2$, then $op(c, E) = (\frac{E}{2}, \frac{E}{2}) = CEL(c, E)$. If $c_1 \neq c_2$, without loss of generality assume $c_1 < c_2$. Let

[27] We define the *order preservation lower bound*, for problems with population size greater than 2, to be equal to zero. Alternatively, we could set it equal to some minimal restrictions imposed by *full order preservation*.

$\lambda(c,E) = \frac{E}{2}$. Then, $op(c,E) = (0, \lambda(c,E))$ and $p^{op}(c,E) = (c_1, c_2 - \lambda(c,E), E - \lambda(c,E))$. Since $E \leq c_2 - c_1$, in the revised problem the endowment is less than the difference of the claims, $E - \lambda(c,E) \leq c_2 - \lambda(c,E) - c_1$. Thus, $op(p^{op}(c,E)) = (0, \lambda(p^{op}(c,E))$. Define $op'(c,E) = op(c,E) + op(p^{op}(c,E))$, then $op'(c,E) = (0, \lambda'(c,E))$, where $\lambda'(c,E) = \frac{3}{4}E$. A recursive argument shows that for some $y \in \mathbb{R}$, $OP(c,E) = (0,y)$, and since OP defines a rule then $y = E$. Thus, $OP(c,E) = (0,E) = CEL(c,E)$.

Now we show that that for each (c,E) with $E > |c_2 - c_1|$ coincidence also holds. Let $N \in \mathcal{N}$ with $|N| = 2$, and $(c,E) \in \mathcal{C}^N$ with $E \geq |c_2 - c_1|$. If $c_1 = c_2$, then $op(c,E) = (\frac{E}{2}, \frac{E}{2}) = CEL(c,E)$. If $c_1 \neq c_2$, without loss of generality assume $c_1 < c_2$. For this problem, *order preservation lower bound* assigns to agent 1 the award assigned to her by the constrained equal losses rule, $op_1(c,E) = \frac{E-(c_2-c_1)}{2} = CEL_1(c,E)$. In the revised problem, $(c', E') = p^{op}(c,E)$, we have $c'_1 < c'_2$ and $E' = c'_2 - c'_1$. Hence, by the previous step, $OP_1(p^op(c,E)) = 0$. By the *recursive* condition $OP_1(c,E) = op_1(c,E) + OP_1(p^{op}(c,E))$. Thus, $OP_1(c,E) = CEL_1(c,E)$. Since the *recursive extension of order preservation* defines a rule $OP_1(c,E) + OP_2(c,E) = E$, hence $OP_2(c,E) = E - OP_1(c,E) = CEL_2(c,E)$. Thus, $OP(c,E) = CEL(c,E)$. □

We are now ready to state our final theorem:

Theorem 3. *The constrained equal losses rule is the only rule satisfying consistency and invariance under the assignment of order preservation lower bound.*

Proof. Let φ be a rule satisfying *invariance under the attribution of order preservation lower bound*. By Theorem 1, for two-agent problems, φ is the unique rule satisfying the recursive extension of *order preservation lower bound*. By Proposition 5, for two-agent problems, such rule coincides with the *constrained equal losses rule*. The *constrained equal losses rule* is *conversely consistent*, and φ is *consistent*. Then, by the Elevator Lemma, for each $N \in \mathcal{N}$, and each $(c,E) \in \mathcal{C}^N$, $\varphi(c,E) = CEL(c,E)$. □

Appendix

Examples of inheritance failure for monotonicity properties.

We provide a two-agent example for *claims monotonicity*. The bound satisfies the property, but its *recursive extension* fails it. For two-agent problems *claims monotonicity* and *others-oriented claims monotonicity* are equivalent. Hence, this example also shows inheritance failure of *others-oriented claims monotonicity*. We provide the rights vectors assigned by the bound and by its recursive extension for selected problems. The bound should be defined for all problems in a way that satisfies the property of interest. In the tables, the problems (c,E) and $(\bar{c},\bar{E})$ are the initial problems and satisfy the monotonicity hypotheses. The problems (c',E') and $(\bar{c}',\bar{E}')$ are the revised problems $p^b(c,E)$ and $p^b(\bar{c},\bar{E})$ respectively.

1. ***Claims monotonicity*** **and** ***others oriented claims monotonicity*****:**

(c_1, c_2, E)	$(30, 90, 100)$	$(\bar{c}_1, \bar{c}_2, \bar{E})$	$(30, 100, 100)$
$b(c_1, c_2, E)$	$(10, 30)$	$b(\bar{c}_1, \bar{c}_2, \bar{E})$	$(10, 45)$
(c'_1, c'_2, E')	$(20, 60, 60)$	$(\bar{c}'_1, \bar{c}'_2, \bar{E}')$	$(20, 55, 45)$
$b(c'_1, c'_2, E')$	$(10, 50)$	$b(\bar{c}'_1, \bar{c}'_2, \bar{E}')$	$(20, 25)$
$B(c_1, c_2, E)$	$(20, 80)$	$B(\bar{c}_1, \bar{c}_2, \bar{E})$	$(30, 70)$
$B(c'_1, c'_2, E')$	$(10, 50)$	$B(\bar{c}'_1, \bar{c}'_2, \bar{E}')$	$(20, 25)$

2. ***Resource monotonicity*:**

(c_1, c_2, E)	$(30, 90, 100)$	$(\bar{c}_1, \bar{c}_2, \bar{E})$	$(30, 90, 110)$
$b(c_1, c_2, E)$	$(10, 30)$	$b(\bar{c}_1, \bar{c}_2, \bar{E})$	$(15, 75)$
(c'_1, c'_2, E')	$(20, 60, 60)$	$(\bar{c}'_1, \bar{c}'_2, \bar{E}')$	$(15, 15, 20)$
$b(c'_1, c'_2, E')$	$(20, 40)$	$b(\bar{c}'_1, \bar{c}'_2, \bar{E}')$	$(10, 10)$
$B(c_1, c_2, E)$	$(30, 70)$	$B(\bar{c}_1, \bar{c}_2, \bar{E})$	$(25, 85)$
$B(c'_1, c'_2, E')$	$(20, 40)$	$B(\bar{c}'_1, \bar{c}'_2, \bar{E}')$	$(10, 10)$

3. ***Population monotonicity:***

(c_1, c_2, E)	$(30, 90, 100)$	$(\bar{c}_3, \bar{c}_1, \bar{c}_2, \bar{E})$	$(10, 30, 100, 100)$
$b(c_1, c_2, E)$	$(10, 30)$	$b(\bar{c}_3, \bar{c}_1, \bar{c}_2, \bar{E})$	$(5, 5, 10)$
(c'_1, c'_2, E')	$(20, 60, 60)$	$(\bar{c}'_3, \bar{c}'_1, \bar{c}'_2, \bar{E}')$	$(5, 25, 80, 80)$
$b(c'_1, c'_2, E')$	$(10, 50)$	$b(\bar{c}'_3, \bar{c}'_1, \bar{c}'_2, \bar{E}')$	$(5, 25, 50)$
$B(c_1, c_2, E)$	$(20, 80)$	$B(\bar{c}_3, \bar{c}_1, \bar{c}_2, \bar{E})$	$(10, 30, 60)$
$B(c'_1, c'_2, E')$	$(10, 50)$	$B(\bar{c}'_3, \bar{c}'_1, \bar{c}'_2, \bar{E}')$	$(5, 25, 50)$

Example of inheritance failure for *invariance under partial assignment of* b.

We provide the rights vectors assigned by the bound and by its recursive extension for 4 problems, for each other problem the bound is set to 0. Well-behaved rules (in terms of monotonicity) can be constructed. Following the notation in the text, let $N = \{1, 2\}$ and $N' = \{1\}$.

(c_1, c_2, E)	$(1, 3, 2)$	$p^b_{\{1\}}(c_1, c_2, E)$	$(1, 2, 1)$
$b(c_1, c_2, E)$	$(.5, 1)$	$b(p^b_{\{1\}}(c_1, c_2, E))$	$(.5, .5)$
$p^b(c_1, c_2, E)$	$(.5, 2, .5)$	$p^b(p^b_{\{1\}}(c_1, c_2, E))$	$(.5, 1.5, 0)$
$b(p^b(c_1, c_2, E))$	$(.1, .4)$	$b(p^b(p^b_{\{1\}}(c_1, c_2, E)))$	$(0, 0)$
$B(c_1, c_2, E)$	$(.6, 1.4)$	$B(p^b_{\{1\}}(c_1, c_2, E)$	$(.5, 1.5)$

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